LOGIC AND METHODOLOGY

OF SOCIAL SCIENCES

2021-22 EDITION

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FIRST PARTIAL

CHAPTER 1: MATHEMATICAL REASONING

Proofs are arguments:

- arguments we write to convince that the statement we are interested in is true
- Goal: prove that it follows from a given set of assumptions and a set of rules for reasoning (logic)

A proof is not to be taken as a proof of its truth but rather as a proof that the sentence is a logical consequence of the hypothesis used in its proof

• you can't proof the truth of anything unless you assume the proof of something (basic definitions and theorems, etc.)

Informal proof - arguments aimed at establishing that a given statement is a consequence of a given set of hypothesis

• each intermediate statement is either taken as obvious or obtained by logical deduction

◦ Taken as obvious

◦ Consequence of a general mathematical knowledge (theorems)

◦ Consequence of explicitly stated hypothesis or statement established with a proof or assumed to be true

• It involves sub-chains of logical deductions

Formal proof - finite sequence of mathematical statements

• such that each element of the sequence is either

◦ the hypothesis

◦ an axiom

◦ the conclusion, obtained from applying the rule of interference to a previous step in the sequence

Rules of Inference: simply logic

- How you get from one step to the next in a proof
- Arguments that are established as valid and therefore can be used without proving

Formal proofs link logic to algorithmic reasoning

• If we had an algorithm that translate informal proofs in formal ones then we wouldn't need human beings anymore

Correlation does not mean causation

We have to prove if a conclusion is an acceptable conclusion from the premises and not if it's true (two different concepts)

• In rhetorical arguments often you get confused with difficult arguments and they end up by saying a simple thing which you understand and you confuse understanding with truth

Mathematical argument - finite sequence of mathematical statements

• mathematical statements - a well-formed linguistic expression for which you can think of a decisive experiment that tells you whether it is true or false (answers yes/no)
  ◦ The experiment unambiguously settles the question: this captures the essence of mathematical statements
• Natural language is inevitably ambiguous - mathematical reasoning tries to get rid of ambiguity by the greatest possible extent

Mathematical reasoning never takes sentences in isolation - if it did, it would be of no interest... instead, it takes into consideration the relations among statements (if...then)

Terminology

• Theorem - a very important result which adds to the stock of knowledge about a specific subject
• Lemma - a preliminary result which makes the actual theorem much easier to read
• Proposition - important result which often lists properties of a mathematical subject
• Corollary - a noteworthy logical consequence following a proved result, usually they don't get named

Consensus paper - document which confirms a statistical trial after having accumulated a sufficient amount of data

Connectives:

- $p \land q$ – P AND Q
- $p \lor q$ – P OR Q
- $p \rightarrow q$ – IF P THEN Q
- $\neg q$ reads - IT IS NOT THE CASE THAT or NOT Q

Notation - if p is "true" - v(p)=1 (the truth value of p is 1)

- If something is the case - antecedent
- Then something is the case - consequent

Procedure for formalizing sentences:

• (1) Read carefully the English statement/ argument (extract the essence of the information you're given)
  ◦ Modeling is about making choices - if it wasn't it would be the same of the reality (find right resolution, as maps JP)
• (2) Identify "the smallest" linguistic expressions which pass the test for being mathematical statements (no connectors)
• (3) You assign distinct letters to them (e.g. p, q, r...)
• (4) Introduce connectives as appropriate

The negation of a true statement is a false statement - the negation of a false statement is a true statement

- Implication with a true premise and false conclusion —> bad reasoning

Formal modelling (of a scientific phenomena) always involves a trade-off between mathematical convenience and the adherence to the phenomenon itself.

- Good mathematical modeling keeps track of assumptions of this kind

The key logical fact about mathematical reasoning is that the truth-value of "compounded sentences" is uniquely determined by the truth-values taken by its components as indicated by the Boolean tables

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
0	0	1	0	0	1	1
0	1	1	0	1	1	0
1	0	0	0	1	0	0
1	1	0	1	1	1	1

TABLE 1.3.1. Boolean tables for common connectives

Counterexample - true premises and false conclusions
Prove by contradiction - negating the conclusion

Implication: p → q can be read as

- If p, then q
- P implies q
- P is a sufficient condition for q
- Q is a necessary condition for p
- Q if p
- P only if q
- Q given p
- Q whenever p
- And implication is evauluated to 1 in exactly one case

Types of proofs:

• "Contrapositive" - ¬q → ¬p - assume consequence is false and derive the antecedent is also false (proof by contraposition)

• Proof by Contradiction: negation of the implication ¬(p → q)

• Converse implication (the most used in mathematical proofs, "conversely"): q → p

Experimental evidence can falsify descriptive theory
Normative theories can only be dismissed on theoretical or epistemological grounds

Mathematical statements must have the domain of quantification made explicit

• Quantifiers - extremely useful and sometimes subtle

- Transitivity: ∀ x,y,z ∈ X if x Pi y and y Pi z then x Pi z [Negation - ∃ a,b,c ∈ X s.t. (a Pi b) and (b Pi c) but not (a Pi c)]
- Order matters:
  - ∃ y s.t. ∀x x<y (FALSE, it is saying that there is a number greater than all other natural numbers)
  - ∀x ∃ y s.t. x<y (TRUE, for every number there is a successor)
- Transitivity: if x is preferred to y and y is preferred to z, then x is preferred to z

Existence of a counterexample is a necessary and sufficient condition to negate (falsify) a universally quantified statement.

Induction principle —> Proof by induction (about recursively defined objects)

- If the statement about (1) holds
- And if the statement about k holds, then the statement about k+1 also holds
- Then: the statement holds for all

Proposition 2 "Principle of induction", suppose - (a) $\theta(1)$ holds, (b) $\theta(k) \to \theta(k+1)$ holds, therefore (c) $\forall k \in N$

Def of N - (a) $0 \in N$ (b) If $n \in N$ then n+1 \in N (c) Nothing else is a natural number

Chapter 2 - ELEMENTARY MATHEMATICAL LOGIC

(1) Introduction of a Formal language - goal is to produce an algorithm to test for what counts as a "mathematical statement"
• In formal language there can be NO ambiguity (you lose some of the precision of the natural language)

(2) Formal semantics - goal is to come up with an algorithm to compute the truth-value of arbitrarily complex mathematical statements from the truth-values of their components

(3) Definition of what it means for a mathematical statement to follow logically from other mathematical statements

Formal languages:

- L = {p_1, p_2,...,p_n} - The language, set of "propositional variable" (elementary mathematical statements, letters)
- Elements of L are the smallest units for which it makes sense to ask whether they are true or false (propositional variables)
- Elements of L don't contain any connectives!!!
- can stand for/be short for any elementary mathematical statement - random variables ranging over {0,1} (true or false)

C = {¬,∧,∨,→} - set of propositional connectives (we are talking about propositional logic)

- Respectively: negation, conjunction, disjunction, implication
- The meaning of the connectives is independent of the interpretation of the statements which they connect
- IF AND ONLY IF: conjunction of two implications
  - (if...then) and (if...then)
- UNLESS = IF NOT
  - NOT Q IMPLIES P

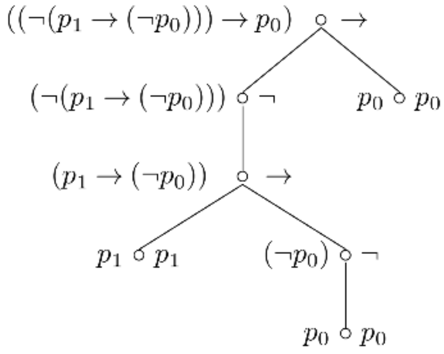
SL - the set of sentences of L, intuitively defined as follows:

- (1) All propositional variables are sentences
- (2) Two sentences with connectives are sentences
- (3) Nothing else is a sentence
- (1) SL_1 = L
- (2) SL_n+1 = SL_n ∪{¬(θ), (θ∗ φ) | θ, φ ∈ SLn,∗ ∈ {∧,∨,→}}
- (3) SL = Infinite Union SL_1 U SL_2 U ... U SL_n
- Lower case Greek letters are elements of SL
- Upper case Greek letters are subsets of SL

Cardinality of L - 2^n where n is the number of propositional variables of L

To visualize sentences we use parsing trees which are as well defined recursively

- The Parsing tree is an algorithm which gives you the language of sentences - look at the leaves and delete repetitions
- Method:
    - Label the first propositional variable
    - Label the main connective
    - Branch out what is to the left and to the right of the main propositional variable



A set is defined recursively when its elements are used to define the set itself

- (1) Base tells you what happens to the first natural number
- (2) Recursive step - suppose you have created this construction up to N and it defines how SL_{n+1} is to be achieved
- (3) Closure step - nothing gets in there unless you apply the provisions of step 2
- Example:
    - 0 is a natural number
    - If n is a natural number, n+1 is also a natural number
    - Nothing else is a natural number

Lemma 1: any sentence in SL has an even number of parantheses

A (alphabet) = { L (language), C (connectives), p}   p={"(", ",", ")"}

- SA "Strings of A" all possible combinations

Unique readability theorem: let θ, φ1, φ2 be elements of SL

- Then exactly one of the following holds :

- θ is in L
- θ= NOT φ1
- θ= φ1 AND φ2
- θ= φ1 OR φ2
- θ= φ1 IMPLIES φ2
- MEANING: there is only one way to construct a sentence so that the sentence is clear and free of any ambiguities
- in SL there are no ambiguities (logical sentences are unambiguous by design)

Conventions – you can omit the outmost parenthesis

Logical meaning:

- no ambiguity is too little ambiguity to be useful - $(p \wedge q) \neq (q \wedge p)$ - syntactically different but same meaning
- only the elements count in sets (no repetition and order) - two sets are the same iff they have the same elements

Syntax - the part of logic to which alphabet and sentences of a language L belong

Semantics - the part of logic that discusses the relations between linguistic objects (sentences) and what is expressed by them

A **valuation** is a function that maps L into a set of truth values v: L -> {0,1}

- 0 is false and 1 is true
- it is a complete description of a logical state of affairs
- the task of logic is not to decide which propositional variables are true but rather drawing logical conclusions from the supposition that the elements are evaluated in a certain way

Lemma 2 - According to truth functionality (aka compositionality) the valuations of L extend uniquely to SL according to the tarskian conditions, given this lemma we can do variables re-naming

- $v(\neg\theta) = 1 \Leftrightarrow v(\theta) = 0$
- $v(\theta \wedge \varphi) = 1 \Leftrightarrow v(\theta) = v(\varphi) = 1$
- $v(\theta \vee \varphi) = 0 \Leftrightarrow v(\theta) = v(\varphi) = 0$
- $v(\theta \to \varphi) = 0 \Leftrightarrow v(\theta) = 1$ and $= v(\varphi) = 0$.
- The truth-value of any sentence is a fixed function of the truth-values of its components
- Generalization of the method of Boolean tables beyond propositional variables - once you have θ, φ you apply the conditions

Extended boolean tables - gives you an algorithm to decided whether any sentence is one of the following

• (1) Tautology iff v(o)=constant 1 (<1,....,1>) - example $p \vee \neg p$ – always true

• (2) Contradiction iff v(0)=constant 0 (<0,...,0>) - example $p \wedge \neg p$ - never true

• (3) Contingent sentence iff it's neither a tautology nor a contradiction

◦ Carbon reduction is incompatible with economic growth "to reduce carbon emissions it is necessary not to grow" $p \to \neg q$

Satisfiability - a truth vector is satisfiable if it has a model (and inconsistent/unsatisfiable otherwise)

- A set is satisfiable if every element of that set has a model (at once)
- satisfiability and associated notion of logical model can be seen as a formalization of the intuitive notion of coherence
- If it is not satisfiable then it is inconsistent - individual rationality leads via majority aggregation to collective irrationality
- To prove satisfiability: take the conjunction of all the sentence of the set → if there is a row when all of them are evaluated to 1, then the set is satisfiable

What does it mean for a set to be consistent? Equivalent to asking if it satisfiable (coherence ~ rationality)

- A set is satisfiable if it has a Model - consistency implies satisfiability

Consistency = satisfiability

Procedure to find the valuations which satisfy a truth vector

- Lexicographic order - forget about the valuations that gives 0, we are only interested in the valuations that give one
    - Stick an ∧ between the results of the valuation and an ∨ between all the possible evaluations

Summary for the first part

Mathematical arguments - finite sequence of mathematical statements (shortest sentences for which you can think of a decisive experiment, named with letters - trying to get rid of ambiguities by the greatest possible extent)

Theorem - important result which adds to the stack of knowledge about a specific subject
Lemma - preliminary result which makes the theorem much easier to read
Proposition - important result which usually lists the properties of a mathematical subject
Corollary - noteworthy logical consequence following from a proved result, usually they don't get named

Procedure of translation - (1) read carefully (2) identify the shortest sentences which can be translated into YES/NO questions (3) assign distinct letters (propositional variables) (4) introduce connectives as appropriate

- in formal modeling there is a trade-off between mathematical convenience and adherence to the phenomenon itself

The language L is the set of propositional variables - it extends uniquely to SL according to the Tarskian conditions (L2, C aka tf)

- a valuation is a function which maps L to the set of truth values {0,1}

Syntax - the branch of logic to which alphabet and language of L belong
Semantics - the branch of logic which discusses the relationship between linguistic objects and their meaning

Tautology - all truth values are 1 // Contradiction - all truth values are 0 // Contingent - neither tautology nor contradiction

Satisfiability - satisfiable when it has a model, unsatisfiable otherwise (tight to the intuitive notion of coherence)

Finding the valuations which satisfy a truth vector - Boolean, delete 0s, write remaining v using ∧ and stick ∨ between them

Second partial

REST OF CHAPTER 2

Logic: Boolean tables provide the key ingredient of algorithms to decide the VALIDITY of argument
Methodology: Validity is necessary but not sufficient to "rational decisions/actions"

- it is not sufficient because two extra-logical features must be checked
- (1) the formalization of the argument must be adequate
- (2) the premises of the argument must be true - empirical knowledge, as long as checking is possible

Syntax - in this setting there is no linguistic ambiguity (unique readability system)
Semantics - valuations to define the meaning of connectives (provided by the boolean tables)

- key point - compositionality aka truth funcitonality (valuations extend uniquely to SL, via the tarskian conditions)
  - The truth value of any sentence is determined by the function of the truth value of its component
  - Extended boolean tables provide a method to compute the truth-vector of any mathematical statement

Logical consequence: We say that θ is a logical consequence of Γ, written $\Gamma \models \theta$ if and only if $\forall$ v∈V, if v(Γ)=1 then v(θ)=1.

- θ is a logical consequence of Γ if and only if every model of Γ is a model of θ, which we write as follows

$$\Gamma \models \theta \Leftrightarrow M\Gamma \subseteq M\theta.$$

- Tip to check if θ is a logical consequence of Γ: check whether all the rows assigning 1 to Γ also evaluate θ to 1
- Possessing informations about logical consequences means excluding all those valuations logically incompatible with them
- Proposition 3. Let $\theta,\varphi \in SL$. Then:
  (1) $\Gamma,\varphi \models \theta \Longleftrightarrow \Gamma \models (\varphi \to \theta)$
  (2) $\theta \models \neg\varphi \Longleftrightarrow \not\exists v \in V$, s.t. $v(\theta)=v(\varphi)=1$
  (3) $\theta \models \varphi \Longleftrightarrow \neg\varphi \models \neg\theta$
  (4) $\emptyset \models (\theta \vee \neg\theta)$
  (5) $\Gamma \models (\theta \wedge \neg\theta) \Longrightarrow \Gamma \models \alpha \ \forall \alpha \in SL.$
  (6) $\theta \models \varphi \Longrightarrow \theta,\psi \models \varphi,\ \forall \psi \in SL$

Remark 13: Proposition 3 makes the connection between "$\to$" and "$\models$"

$$\varphi \models \theta \Longleftrightarrow \models \varphi \to \theta$$

Remark 14: $\Gamma \models (\theta \wedge \neg\theta) \Rightarrow Mod(\Gamma) \subseteq Mod(\theta \wedge \neg\theta)$
$\Rightarrow Mod(\Gamma) \subseteq \emptyset$
$\Longrightarrow Mod(\Gamma) = \emptyset$
$\Longrightarrow \emptyset \subseteq Mod(\alpha)$, for any $\alpha \in SL$

Tautology: when all the rows of the Boolean table are evaluated to 1
Contradiction: a sentence that has no models (it is always evaluated to 0)

Logical Equivalence: θ, φ ∈ SL are logically equivalent, writ- ten θ ≡ φ, if ∀v ∈ V, v(θ) = v(φ).

- Equivalent definitions of logical equivalence: θ≡φ
  - ⇐⇒ ∀v ∈ V, v(θ) = v(φ)
  - ⇐⇒∀v∈V,v(θ)=1⇔v(φ)=1
  - ⇐⇒∀v∈V,v(θ)=0⇔v(φ)=0
  - ⇐⇒ ∀v ∈ V, v(θ) = 1 ⇒ v(φ) = 1 and ∀v ∈ V, v(φ) = 1 ⇒ v(θ) = 1
  - ⇐⇒ θ |= φ and φ |= θ
  - ⇐⇒|=θ→φ and |=φ→θ
  - ⇐⇒ ∀v ∈ V, v(θ → φ) = 1 and v(φ → θ) = 1
  - ⇐⇒ ∀v ∈ V, v((θ → φ) = 1 and v(φ → θ)) = 1
  - ⇐⇒|= (θ → φ) ∧ (φ → θ)
- BASIC PRINCIPLES:
  - Reflexivity: φ ≡ φ
  - Symmetry: If θ ≡ φ then φ ≡ θ
  - Transitivity: If θ ≡ φ and φ ≡ ψ then θ ≡ ψ.

CHAPTER 3: APPLYING LOGIC

There exists an algorithm to decide, for θ ∈ SL and Γ ⊆ SL, whether or not **Γ |= θ**

- Express the sets as p,q,r, etc.
- Construct the respective Boolean table
- Check whether the models of Γ is a subset of the models of θ
- Bottom line: check whether the models of what is before the logical equivalence sign is included in the models of what is after

Mathematical modeling is necessary not sufficient to make right choices - arguments may be sound but not decision-relevant

• Modeling may not be adequate - you have an algorithm to decide validity but you have to question the inputs

• Logical enquiry can only determine whether a certain sentence is a logical consequence of a given set of premises

First-order logic **"Logic of quantifiers" =** logic of quantifiers (\forall \exists)
Propositional logic = connectives

|= rests entirely on the semantical notion of satisfaction and the related notion of a model of a sentence
• those notions rest on the fundamental concept of propositional valuation

Propositional arguments - arguments which can be translated into propositional logic

• A propositional argument is decision-relevant in a given context if (1) it is valid and (2) all its premises are satisfied

◦ A propositional argument is a claim that a sentence θ follows from a given set of premisses {γ_1, . . . γ_n}

• We say that the argument is correct/valid/sound if and only if θ is a logical consequence of the conjunction of all γ

- Decision-relevance is not a formal requirement, cannot be determined by logical reasoning – rather, general information about the context is helpful in deciding whether the premises are true
- There are several logically equivalent ways to translate the same argument
- Notation for premises versus conclusions: fraction line or the three dots

$$p$$
$$p \to q$$
$$\therefore\ q$$

The language of propositional logic is not adequate to carry out all valid arguments in elementary number theories

Discharged premise: a premise that is not needed when making the decision, it makes no difference whether it is there or not
Undischarged premise = relevant assumption

For an argument to be valid...

- The formalization must be accurate (this is subjective)
- Boolean tables can be only used if the argument can be accurately translated into propositional logic's language

Logical Description of the world:

- V^L stands for the 4 rows of the Boolean table if L=2 variables
- The 4 rows represent the description of the world when we have no information about it (otherwise, rows could be ruled out)
- For example: possessing information s.t. θ|=φ means excluding all the valuations that are logically incompatible with it (e.g. excluding when v(θ)=1 and v(φ)=0)

Logical Independence: propositional variables in L are logically independent

p	q	Case excluded gives
V1 0	0	subcontraries
V2 0	1	q→p
V3 1	0	p→q
V4 1	1	contraries

The set of atoms is the set of maximal consistent conjunctions of literals

• Maximal - every propositional variable is shown in the atoms (exactly one is true!)
• Consistent - p and not p never occurs
• Conjunction of literals - either a propositional variable or the negation of a propositional variable

ATL ={¬p∧¬q,¬p∧q,p∧¬q,p∧q}

- If one of the atoms are satisfied, we know the rest are not
- No valuation (row) satisfies more than one atom

Claim 2: Let L = {p,q}. Then any θ ∈ SL can be written as a disjunction of its atoms

- For instance: if p→q is the formalization of the sentence, v_3 can be excluded , so we can write it as the conjunction of all the atoms except i=3
- E.g. If we have the truth values: θ = ⟨0,1,0,0,0,0,0,1⟩
  - We know that θ = (Atom2 AND Atom8)

- All sentences from a finite language can be expressed as the disjunctions of conjunction of literals

If you substitute a sentence with a logically equivalent sentence, the validity of the argument is preserved

PROPOSITION 5. *The following hold*

$$v\left(\bigvee_{i=1}^{n}\theta_i\right) = 0 \Leftrightarrow v(\theta_i) = 0,\ for\quad i = 1,\dots,n \qquad \text{(i)}$$

$$\bigvee_{i=1}^{n}\theta_i \equiv \neg\bigwedge_{i=1}^{n}\neg\theta_i. \qquad \text{(ii)}$$

Each element of SL is uniquely identified by the symbols that make it up = each sentence is syntactically unique. BUT, each sentence can be expressed by at least another sentence (they are logically equivalent).
→ Syntactic differences between logically equivalent sentences do not matter

PROPOSITION 8. *Let $\vec{1}$ denote any tautology and similarly $\vec{0}$ for contradictions. The following hold:*

$$\theta \wedge \vec{1} \equiv \theta \qquad \text{(Identity for conjunction)}$$
$$\theta \vee \vec{1} \equiv \vec{1} \qquad \text{(Identity for disjunction)}$$
$$\theta \wedge \vec{0} \equiv \vec{0} \qquad \text{(zero for conjunction)}$$
$$\theta \vee \vec{0} \equiv \theta \qquad \text{(zero for disjunction)}$$

PROPOSITION 9. *The following hold:*

$$\neg\neg\theta \equiv \theta \qquad \text{(Inverse)}$$
$$\theta \wedge \theta \equiv \theta \qquad \theta \vee \theta \equiv \theta \qquad \text{(Idempotence)}$$
$$\theta \wedge \varphi \equiv \varphi \wedge \theta \qquad \theta \vee \varphi \equiv \varphi \vee \theta \qquad \text{(Commutatitivy)}$$
$$\theta \wedge (\varphi \wedge \psi) \equiv (\theta \wedge \varphi) \wedge \psi \qquad \theta \vee (\varphi \vee \psi) \equiv (\theta \vee \varphi) \vee \psi \qquad \text{(Associativity)}$$

PROPOSITION 10. *The following hold:*

(1) Distributivity

$$\theta \vee (\varphi \wedge \psi) \equiv (\theta \vee \varphi) \wedge (\theta \vee \psi) \qquad (51)$$
$$\theta \wedge (\varphi \vee \psi) \equiv (\theta \wedge \varphi) \vee (\theta \wedge \psi) \qquad (52)$$

(2) De Morgan

$$\neg(\theta \wedge \varphi) \equiv (\neg\theta \vee \neg\varphi) \qquad (53)$$
$$\neg(\theta \vee \varphi) \equiv (\neg\theta \wedge \neg\varphi) \qquad (54)$$

The "is-ought problem" Hume - when writers make claims on what ought to be that are based solely on what is

- there is a significant difference from positive statements (what is) and normative (what is ought to be) ones
  - It is not obvious how one can move coherently from descriptive statements to prescriptive ones
- Hume's law - a reasoner has access to non-moral factual premises, he cannot logically interfer the truth of moral statements

CHAPTER 4: LOGIC AND PROBABILITY

Definition of a Model: $[\theta] = \{v \in V \mid v(\theta) = 1\}$

PROPOSITION 12:

(1) $[\theta \vee \varphi]=[\theta] \cup [\varphi]$ – the probability of θ AND φ equals the union of the probability of θ and the probability of φ
(2) $[\theta \wedge \varphi]=[\theta] \cap [\varphi]$ – the probability of θ OR φ equals the intersection of the probability of θ and the probability of φ
(3) $[\neg\theta] = V - [\theta]$, i.e. the complement of [θ] with respect to V - the probability of NOTθ is the complement of the probability of θ

One definition of Probability: $P(\psi) = Card([\psi])/Card(V)$ (the ratio between the cardinality of its models to the cardinality of all the models possible)

- any probability function is a convex combination of truth-values
- if two sentences are logically equivalent, they have equal probability
- probability functions are not compositional (compositionality means that the value of an expression is determined by the values of its elements and the rules used to combine them)

Definition of **Probability Functions:** θ, φ are in SL, then P: SL → [0,1] is a probability function if it satisfies:

- If $\models \theta$ then $P(\theta) = 1$
- If $\models \neg(\theta \wedge \varphi)$ then $P(\theta \vee \varphi)=P(\theta)+P(\varphi)$.
- Some atoms can be excluded (they have 0 probability)

Representation Theorem: ATL ∋αi →ai ∈[0,1]

- all the atoms of a set have a probability between 0 and 1
- the sum of the probability of the atoms must equal 1
- probability arises from logic

Probability: Even if the argument is valid, and the model is adequate, your answer may not be objective

• Omega (event)= {w_1,...,w_n} sample space
• F an algebra of omega

Probability presuppose propositional logic - we do not have an algorithm to compute the probability of compound sentences

• unless very comprehensive and accurate data are available, disagreement on the probability of certain events is the norm

UNDERSTAND REMARK 22 - mathematical formalization is necessary because if your model is not consistent there is no use you can make of it (once you have proved it is specific knowledge is required to know if it addresses your problem)

CHAPTER 6 : LOGIC AND RATIONALITY
(LOGIC AND THE SOCIAL SCIENCES)

Rationality: necessary to understand many aspects of behavior, BUT humans don't always act rationally

- characterizes the individual (decision theory)
- interactions (game theory)
- collective groups (social choice theory)
- either humans behave randomly or they don't → if they don't behave randomly, patterns can be found in their behavior

Why Logic?

- Appeal to a notion of coherence/satisfiability/consistency: a set of sentences is satisfiable if there exists a valuation which maps all of its elements into 1
- If a set is satisfiable, it is "possible" (it can happen)
- Avoiding cyclic preferences (where you end up with where you started, but with less money) is a necessary condition for avoiding irrational behavior

Rationality in microeconomics means that the preference relationship is consistent (=coherent, satisfiable)

• Logic can be applied to social-scientific models as follows :
  ◦ (1) check that the model make for a satisfiable set
  ◦ (2) explore its logical consequences

• Predicting deviations from the norm is one central aim of behavioral social sciences

Descriptive theory: the goal is to describe the behavior of something (e.g. consumers)
  • experimental evidence leads to possible refutation

Normative theory: the goal is to define what cosumers should do, they are not open to experimental refutation

Rationality as Coherence:

- Not all relations will make sense in the specific context we are in
- Axioms for preference rule out relations that represent irrational behaviors
- Providing justification for rejecting a normative principle (objective principle) is necessary
- Whether an argument is decision-relevant goes beyond logic
- Mathematical formalization is necessary to achieve consistency, but it is not enough to provide one unique answer
- Sufficient: enough, necessary: needed

From individual rationality to a rational society:

- The rule for determining the collective preference can be expressed as a fixed function of the individual preferences – compositionality
- Is there an algorithm that takes the consistent inputs and outputs the consistent preferences (collective preferences)? No, because a majority aggregation of individual consistent preferences can lead toa. Collectively inconsistent one
- Majority is always inconsistent!

Logical Arguments: all the models of the premises must also be models of the conclusion

- This eliminates the possibility of disagreeing on the validity of an argument
- If two individuals disagree on the conclusion of an argument, either: one of them is wrong OR the premises are inconsistent
- Rational disagreement is possible, and frequent

Democracy:

- Truth is a hard concept to define properly, but in a formalized language, truth is (1) formally correct, (2) materially adequate (satisfactory)
- Conditions of material adequacy by Arrow:
    - Unrestricted domain: individuals can have preferences over a whole set – all choices are allowed, no single choice is ruled out
    - Non dictatorship: there doesn't exist a person whose preferences are enough to determine the preference of the society
    - Pareto: if everyone in the society prefers x to y, the society cannot prefer y to x collectively
    - Independence of irrelevant alternatives: when considering social preferences between x and y, preferences for other alternatives is irrelevant – only the individuals preferences on what is decided upon are important
- The notional of formal consistency and democracy cannot be satisfied - you have to give up something (dictatorship is the only logically consistent form of government)
- Discursive dilemma - from individual rationality you can move to collective irrationality with democracy
- Logic and majority are inconsistent - either the majority rule or propositional logic must be given up

If someone prefer less money to more money it is blatantly irrational (Money-pump argument)

Collective Preferences:

- Suppose your society is finite and the individual have consistent individual preferences
- Is this sufficient to define algorithmically the preferences of the society as a whole? (Starting from the preferences of the individuals)
- Compute uniquely what's best with society starting from the aggregation of the individuals' preferences
- → This is not the case: if it was we would just solve a large group of problems

Discursive dilemma: from individual rationality you can move to collective irrationality with democracy

Judgement aggregation:

- We would consider highly democratic a society in which the people want something, and the society doesn't do it - this is incompatible with logic
- Logic and majority are inconsistent - either the majority rule or propositional logic must be given up

Relevant questions

Chapter 1

What are mathematical arguments?

What about mathematical statements? how should you treat ambiguities (non-obvious cases)?

What is a theorem?

What is a Lemma?

What is a proposition?

What is the procedure to follow when translating an English sentence into a mathematical argument?

What is the trade-off in formal modeling?

What is the contrapositive of p implies q? What does a counterexample negate?

What is the difference between descriptive and normative statements?

What is needed in a mathematical statement in order to be interpretable?

Chapter 2

How is the language defined and what composes it?

How is SL defined?

What is the cardinality of L?

How is a recursive definition structured?

How many parentheses does a formal sentence in logic need to have?

What is the unique readability theorem?

Do repetition and order count when you're defining sets?

What is syntax?

What is semantics?

What is valuation?

What is Lemma 2?

Which types of sentences can we define and what do we use to know which one is which?

When is a truth vector satisfiable? What are satisfiability and the notion of logical model associated to?

How do you find which valuations satisfy a truth vector?

Chapter 3

What is the path to construct arguments which lead to rational decisions?
• Boolean tables for validity —> (1) formalization must be adequate and (2) premises of the argument must be true
◦ Logical enquiry can only determine whether a sentence is a logical consequence of a set of premises

What are the quantifiers of logic?

What is a discharged premise?

How do you define the atoms of a set?

What are the two connectives that any sentence can be written with (possibly not in the shortest way, but still)?

What are propositional arguments?
• arguments which can be translated into propositional logic - a claim that θ follows from a given set of premises {γ1, . . . γn}
• when is it decision relevant in a given context? (1) it is valid and (2) all its premises are satisfied

Relationship between soundness (=validity) and decision relevance?
• Decision-relevance implies validity, the opposite does not hold

When is an argument correct or valid or sound?
• if and only if θ is a logical consequence of the conjunction of all γ

Is the language of propositional logic adequate to carry out all valid arguments in elementary number theory?
• no, you would need First order logic+arithmetic

What do you do with information about logical consequences?
• they let you exclude all those valuations logically incompatible with them

Do arbitrary substitutions of logical equivalents impact validity?
• no, validity is preserved

Difference between propositional logic and probability?
• We do not have an algorithm to compute the probability of compound sentences - rational disagreement is the norm

What is the set of atoms?
• it is the set of the maximal (every propositional variable) consistent (no p and not p) conjunction of literals

What is the “is-ought problem” of Hume about?
• significant difference between positive and normative statements - not obvious how one can move coherently from descriptive statements to prescriptive ones (from factual premises you cannot logically interfere the truth of moral statements)

Chapter 4

What is the definition of probability?

How do you compute probability?

What does it mean for a set to be consistent?
• Equivalent to asking if it satisfiable (coherence ~ rationality, a set is satisfiable if it has a model)
• Formal consistency imply the existence of a model (and hence satisfiability)

What is rationality in microeconomics?
• It means that the preference relationship is consistent (=coherent, satisfiable)

How can Logic be applied to social-scientific models? (And how can normative statements be challenged)
• (1) check that the model make for a satisfiable set and (2) explore its logical consequences
• Predicting deviations from the norm is one central aim of behavioral social sciences

What is a descriptive theory?
• The goal is to describe the behavior of something (e.g. consumers) and experimental evidence can lead to refutation

What is a normative theory?
• The goal is to define how something is ought to be (e.g. what consumers should do), not open to experimental refutation

What does it mean that in a valid argument there is no "loss of satisfiability"?
• the consequence is never "less true" than its antecedent
• There cannot be any drop of satisfiability when the premises are evaluated to 0 ???

What are you looking for when checking the validity of an argument?
• whether or not the conclusion is a logical consequence of its premises

Are probability functions compositional?
• always compositional with respect to its negation
• you cannot derive the probability of a compound argument from the probability of its components

Chapter 6

When does satisfiability coincide with consistency? When we are working with a complete deductive system

If a model is consistent then it is satisfiable
• Inconsistency = unsatisfiability

4 conditions identified by Arrow of material adequacy for any aggregation method to reflect individual preferences? UNPI
• Unrestricted domain - there is not a priori preferences (the society wouldn't be democratic if some choices weren't allowed)
• Non dictatorship - agent whose preference is sufficient to determine the preference of the society
• Pareto - if the individuals have unanimous preferences then the society has the same preference
• Independence of irrelevant alternatives - only the individual preferences on what's being decided are relevant to the decision

What does it mean to check the validity of an argument?
• checking whether the conclusion of an argument is a logical consequence of its premises

When does satisfiability coincide with consistency? When we are working with a complete deductive system
If a model is consistent then it is satisfiable
• Inconsistency = unsatisfiability

Which connectives are sufficient to write any sentence? Disjunction (or conjunction) and negation

IN COLLABORAZIONE CON

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