



**MATHEMATICS (MODULE 1)**  
**FIRST PARTIAL**  
**NOTES**

**A.Y. 2023 - 2024**

*Written by Dzianis Rabchuk*

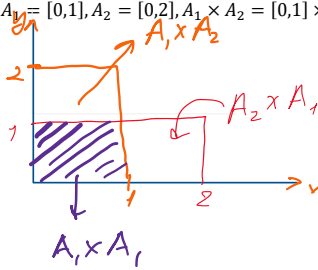
Main formulas	Homework
	Chapter 2 - exercises Proofs in blue

Def:  $A_1, A_2$  subsets of  $R$ . The set of all the ordered pairs  $(a_1, a_2)$  indicated with  $A_1 \times A_2$ , defined by  
 $A_1 \times A_2 := \{(a_1, a_2) : a_1 \in A_1 \text{ and } a_2 \in A_2\}$   
 Is called the Cartesian product of  $A_1, A_2$

E.g.  
 $A_1 = \{1,2\}, A_2 = \{2,3\}, A_1 \times A_2 = \{(1,2), (1,3), (2,2), (2,3)\}$   
 $A_2 \times A_1 = \{(2,1), (2,2), (3,1), (3,2)\}$

In general,  $A_1 \times A_2 \neq A_2 \times A_1$

Eg.  
 $A_1 = [0,1], A_2 = [0,2], A_1 \times A_2 = [0,1] \times [0,2]$

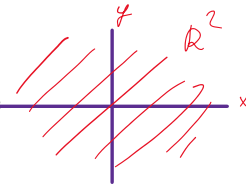


$$A_1 \times A_2 = \{(x,y) : x \in [0,1] \text{ and } y \in [0,2]\}$$

$$A_2 \times A_1 = \{(x,y) : x \in [0,2] \text{ and } y \in [0,1]\}$$

NB.

$A_2 = A_1 = R$ , then  $A_1 \times A_2 = A_2 \times A_1 = R \times R = R^2$   
 $R^2$  set of the points of a plane  
 $R^2 = \{(x,y) : x \in R \text{ and } y \in R\}$



Def:  
 $A_1, A_2, \dots, A_n$  subsets of  $R$ . The set of all the ordered n-tuples  $(a_1, a_2, \dots, a_n)$ , indicated with  $\prod_{i=1}^n A_i$  or  $\prod_{i=1}^n A_i$   
 defined by  
 $\prod_{i=1}^n A_i = \{(a_1, a_2, \dots, a_n) : a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n\}$   
 Is called the Cartesian product of  $A_1, \dots, A_n$

N.B.  $A_1 = A_2 = \dots = A_n = R$  then

$$\prod_{i=1}^n A_i = \prod_{i=1}^n R = R \times R \times \dots \times R = R^n$$

Which is called the **n-dimensional** Euclidean space  
 ....

An element of  $R^n$  is  $x = (x_1, x_2, \dots, x_n)$  which is called a vector of  $R^n$ .  $x_i$  is called the component with i-th position

Eg

$$x = (-1, 2) \in R^2$$

$$y = (1, 0, 3) \in R^3 \quad (y_1 = 1, y_2 = 0, y_3 = 3)$$

## Operations in $R^n$

Given  $x, y \in R^n$  we can define the following operations

a. **Addition of x and y**

$$\begin{cases} x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ \text{Sum of } x \text{ and } y \rightarrow \text{vector} \end{cases}$$

E.g.  $\begin{cases} x = (-1, 2, 3) \in R^3, y = (2, -1, 0) \in R^3 \\ x + y = (-1, 2, 3) + (2, -1, 0) = (1, 1, 3) \in R^3 \end{cases}$

b. **Multiplication of x by a scalar a**

$$a * x = (ax_1, ax_2, \dots, ax_n) \in R^n$$

E.g.  $x = (2, 1, 0, -2), a = 3, 3x = 3(2, 1, 0, -2) = (6, 3, 0, -6)$

Notations:

i.  $0 = (0, 0, \dots) \in R^n \rightarrow$  zero vector of  $R^n$   
 $0 \in R^n, 0 = (0, 0)$

ii.  $-x = (-1)x = (-x_1, -x_2, \dots, -x_n) \in R^n$   
 $-x$  is said to be the opposite of  $x$

E.g.  $x = (2, 1, -3) \in R^3, -x = (-2, -1, 3)$

iii.  $x - y = x + (-1)y$

Proposition 49  $x, y, z \in R^n$ . We have

*is said to be the neutral element (for addition)*

- 1)  $x + y = y + x$   
 Proof: Let  $x, y$  be vectors of  $R^n$ . Therefore  $x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) = (y_1 + x_1, y_2 + x_2, \dots, y_n + x_n) = y + x$
- 2)  $(x + y) + z = x + (y + z)$   
 Proof: Let  $x, y, z$  be vectors of  $R^n$ .  
 $(x + y) + z = (x_1 + y_1 + z_1, x_2 + y_2 + z_2, \dots, x_n + y_n + z_n) = (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \dots, x_n + (y_n + z_n)) = x + (y + z)$
- 3)  $\forall x \in R^n, x + 0 = 0 + x = x$   
 Proof: Let  $x$  be a vector of  $R^n$ . We have  
 $x + (-x) = (x_1 + (-x_1), x_2 + (-x_2), \dots, x_n + (-x_n)) = (0, 0, \dots, 0) = 0$   
 $x + 0 = (x_1 + 0, x_2 + 0, \dots, x_n + 0) = (x_1, x_2, \dots, x_n) = x$
- 4)  $\forall x \in R^n, \exists -x \in R^n, x + (-x) = (-x) + x = 0$   
 $-x = (-x_1, -x_2, \dots, -x_n)$

Proposition 50. Let  $x, y$  be vectors of  $R^n$  and  $a, b \in R$

1)  $a(x + y) = ax + ay$

Proof: Let  $x, y$  be vectors of  $R^n$  and  $a \in R$

$a(x + y) = (a(x_1 + y_1), a(x_2 + y_2), \dots, a(x_n + y_n))$   
 $= (ax_1 + ay_1, ax_2 + ay_2, \dots, ax_n + ay_n)$   
 $= (ax_1, ax_2, \dots, ax_n) + (ay_1, ay_2, \dots, ay_n) = ax + ay$

2)  $(a + b)x = ax + bx$

Proof: Let  $x, y$  be vectors of  $R^n$  and  $a, b \in R$

$(a + b)x = ((a + b)x_1, (a + b)x_2, \dots, (a + b)x_n)$   
 $= (ax_1 + bx_1, ax_2 + bx_2, \dots, ax_n + bx_n) = x \cdot 1 = (x_1, x_2, \dots, x_n) = x$   
 $= (ax_1, ax_2, \dots, ax_n) + (bx_1, bx_2, \dots, bx_n) = ax + bx$

3)  $1 * x = x * 1 = x$

$a(bx) = a(bx_1, bx_2, \dots, bx_n) = (abx_1, abx_2, \dots, abx_n) = (ab)(x_1, x_2, \dots, x_n) = (ab) \cdot x$

4)  $a(bx) = (ab)x$

$\square$  It is said to be the neutral el. of the operat. of mult. by scalar

### Inner or scalar product of vectors:

Def.  $x, y \in R^n$ , The real number, indicated with  $x \cdot y$ , defined by

$$x \cdot y = x_1 * y_1 + x_2 * y_2 + \dots + x_n * y_n = \sum_{s=1}^n x_s * y_s$$

is called the inner product of  $x$  and  $y$

Eg.

$x = (-1, 2, 3), y = (-1, -1, 2)$

$x \cdot y = \sum_{s=1}^n x_s * y_s = (-1)(-1) + 2(-1) + 2 * 3 = 1 - 2 + 6 = 5$

Eg.

$x = (a, -1, 0), y = (a, 4, 2)$

$x \cdot y = \sum_{s=1}^3 x_s * y_s = a * a + (-1) * 4 + 0 * 2 = a^2 - 4$

Find  $a$  such that  $x \cdot y = 0$ :

$x \cdot y \Rightarrow a^2 - 4 = 0 \Rightarrow a = \pm 2$

### Proposition 51

$x, y, z \in R^n$  and  $a \in R$ . We have

- $x \cdot y = y \cdot x$
- $(x + y) \cdot z = x \cdot z + y \cdot z$
- $ax \cdot y = a(x \cdot y)$

NB.  $x, y, z \in R^n$  and  $a, b \in R$ . We have

$(ax + by) \cdot z = a(x \cdot z) + b(y \cdot z)$

Eg.

$x, y, z \in R^3$  and  $x \cdot z = 10, y \cdot z = -1$

$(2x - 3y) \cdot (2z) = (2x) \cdot (2z) + (-3y) \cdot (2z) = 4(x \cdot z) - 6(y \cdot z) = 4 * 10 - 6(-1) = 46$

### Order structure of $R^n$

NB.  $R$  is an ordered set with respect to the relation ' $\leq$ ':  $\forall x, y \in R$  we have either  $x \leq y$ , or  $y \leq x$ . ( $R$  is an ordered set)

E. g.

a.  $x = (2, 1)$  and  $y = (1, 0)$

$x_1 = 2 \geq 1 = y_1$  and  $x_2 = 1 \geq 0 = y_2$

$x \geq y$  (since it's true for all elements with corresponding positions)

$x_1 = 2 \geq 1 = y_1$  and  $x_2 = 1 \leq 0 = y_2$   
 $x, y$  are **not comparable** vectors of  $R^2$

NB.

In  $R^n (n \geq 2)$  we have a partial order of the elements with respect to the relation ' $>$ '

### 1. Weak inequality $x, y \in R^n$

We write  $x \geq y$  if  $x_i \geq y_i, \forall i \in N$  ( $x, y$  can be equal vectors)

NB.  $x \leq y$  if  $x_i \leq y_i, \forall i \in N$

E.g.  $x = (3, 2, 0), y = (2, 1, 0)$

$3 > 2, 2 > 1, 0 \geq 0$   
 $x \geq y$

NB.  $x = y \Leftrightarrow x \geq y$  and  $x \leq y$

EG.  $x = (a, 1), y = (1, 0), a \in \mathbb{R}$

Are  $x$  and  $y$  comparable vectors of  $\mathbb{R}^2$ ?

$a \geq 1$ :  $x, y$  are comparable vectors and  $x \geq y$

$a < 1$ :  $x, y$  are not comparable

**Properties about the relation " $\leq$ ":**

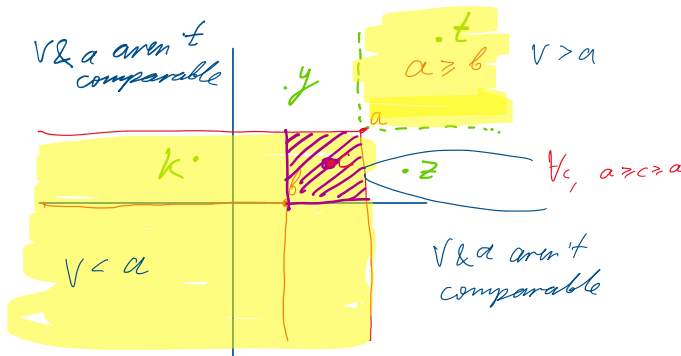
- $\forall x \in \mathbb{R}^n, x \geq x$
- If  $x \geq y$  and  $y \geq z$ , then  $x \geq z$
- If  $x \geq y$ , then  $\forall z \in \mathbb{R}^n, x + z \geq y + z$
- If  $x \geq y$  and  $a \in \mathbb{R}, z \in \mathbb{R}^n$ , then

i. $a * x \geq a * y$ if $a > 0$	i. $x * z \geq y * z$ if $z > 0$
ii. $a * x = a * y = 0$ if $a = 0$	ii. $x * z = y * z = 0$ if $z = 0$
iii. $a * x \leq a * y$ if $a < 0$	iii. $x * z \leq y * z$ if $z < 0$

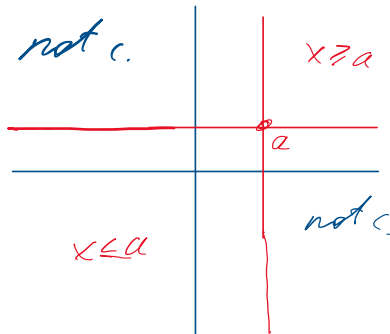
- Separation:**  $A, B \in \mathbb{R}^n$ . If  $\forall a \in A, \forall b \in B, a \geq b$ , then an element  $c \in \mathbb{R}^n$ , exists such that  $\forall a \in A, \forall b \in B, a \geq c \geq b$

EG.

$a = (2, 1), b = (1, 0), a, b \in \mathbb{R}^2$  (comparable vectors)



NB.



## 2. Strict inequality

Let  $x, y$  be vectors of  $\mathbb{R}^2$

$x > y$  if  $x_i \geq y_i, \forall i = 1, 2, \dots, n$ , and a position exists such that  $x_i > y_i$  ( $x$  and  $y$  are always different vectors)

EG.

$$x = (2, 1), y = (-1, 1) \quad \begin{matrix} x_1 = 2 > -1 = y_1 \\ x_2 = 1 = y_2 \end{matrix}, x > y \Rightarrow x \geq y$$

## 3. Strong inequality $x, y \in \mathbb{R}^n$

$x \gg y$  if  $x_i > y_i, \forall i = 1, \dots, n$

EG.

$$x = (2, -1), y = (1, -2), x_1 = 2 > 1 = y_1, x_2 = -1 > -2 = y_2, x \gg y$$

NB.  $x \gg y \Rightarrow x > y \Rightarrow x \geq y$

Given  $x \in \mathbb{R}^n$ , we have

- $x \geq 0$ ,  $x$  is a positive vector
- $x > 0$ ,  $x$  is a strictly positive vector
- $x \gg 0$ ,  $x$  is a strongly positive vector

## Generalization of intervals of $\mathbb{R}$ .

Let  $x, y$  be vectors of  $\mathbb{R}^2$

- Bounded closed interval

$$[a, b] = \{x \in \mathbb{R}^n: a \leq x \leq b\}$$

- Bounded open interval

$$(a, b) = \{x \in \mathbb{R}^n: a \ll x \ll b\}$$

- Bounded half-open / half-closed intervals

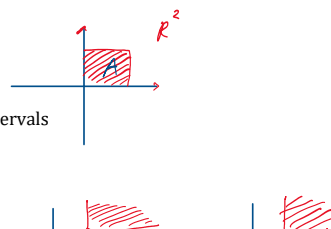
$$(a, b] = \{x \in \mathbb{R}^n: a \ll x \leq b\}$$

$$[a, b) = \{x \in \mathbb{R}^n: a \leq x \ll b\}$$

- Unbounded intervals

$$[a, +\infty) = \{x \in \mathbb{R}^n, x \geq a\}$$

$$(-\infty, a] = \{x \in \mathbb{R}^n, x \leq a\}$$



$$(a, b] = \{x \in \mathbb{R}^n : a < x \leq b\}$$

$$[a, b) = \{x \in \mathbb{R}^n : a \leq x < b\}$$

#### 4. Unbounded intervals

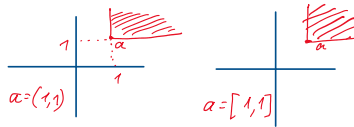
$$[a, +\infty) = \{x \in \mathbb{R}^n, x \geq a\}$$

$$(a, +\infty) = \{x \in \mathbb{R}^n, x > a\}$$

Eg.  $a = (1, 1)$

$$(-\infty, b] = \{x \in \mathbb{R}^n, x \leq b\}$$

$$(-\infty, b) = \{x \in \mathbb{R}^n, x < b\}$$



#### Special cases

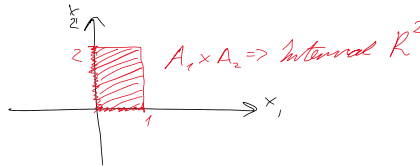
$$[0, +\infty) = \{x \in \mathbb{R}^n : x \geq 0\} = \mathbb{R}_+^n$$

$$(0, +\infty) = \{x \in \mathbb{R}^n : x > 0\} = \mathbb{R}_+^{n,+}$$

$$(-\infty, 0] = \{x \in \mathbb{R}^n : x \leq 0\} = \mathbb{R}_-^n$$

$$(-\infty, 0) = \{x \in \mathbb{R}^n : x < 0\} = \mathbb{R}_-^{n,-}$$

NB. Intervals in  $\mathbb{R}^n$  can be written as Cartesian product of intervals of  $\mathbb{R}$   
 Eg. Let  $\mathbb{R}^2$  be a set of all the points of the plane. Consider  $A_1 = [0, 1]$  and  $A_2 = [0, 2]$   
 $A_1 \times A_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [0, 1] \text{ and } x_2 \in [0, 2]\}$



### Pareto Optima:

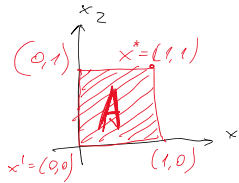
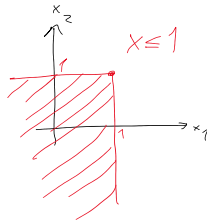
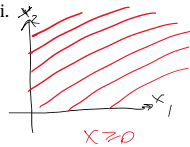
Let  $A$  be a subset of  $\mathbb{R}^n$

- $x^* \in A$  is called a maximum of  $A$  if  $\forall x \in A, x \leq x^*$
- $x_* \in A$  is called a minimum of  $A$  if  $\forall x \in A, x \geq x_*$

Eg.  $A = \{x \in \mathbb{R}^2 : x \geq 0, x \leq 1\}$

Properties:

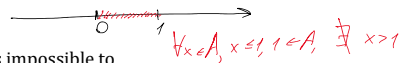
i.



$x^* = (1, 1) \in A$  and  $\forall x \in A, x \leq x^* \Rightarrow x^*$  is a maximum of  $A$   
 $x' = (0, 0) \in A$  and  $\forall x \in A, x \geq x' \Rightarrow x'$  is a min. of  $A$

### Lemma 54

Let  $A \subseteq \mathbb{R}$ . A point  $x^* \in A$  is a maximum of  $A \Leftrightarrow$  no  $x \in A$  such that  $x > x^*$



NB.  $\mathbb{R}$  is a set with total order and in  $\mathbb{R}^n (n \geq 2)$  we have a partial order. It's impossible to generalize Lemma 54 for  $\mathbb{R}^n (n \geq 2)$

EG.

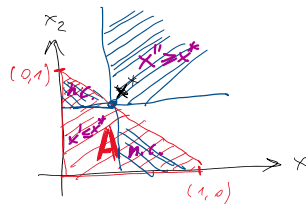
$$A = \{x \in \mathbb{R}^2 : x \geq 0 \text{ and } x_1 + x_2 \leq 1\}$$

$0 \in A$  and  $\forall x \in A, 0 \leq x, 0$  is a minimum of  $A$

$x^* = (\frac{1}{2}, \frac{1}{2}) \in A$  - is not a point of maximum of  $A$

Because  $\exists x \in A : x \not\leq x^*$  (eg  $\tilde{x} = (0, 1)$ )

$x^* = (\frac{1}{2}, \frac{1}{2})$  is said to be a maximal of  $A$  ( $x^*$  is not dominated by any other point of  $A$ )



Def.

Let  $A \in \mathbb{R}$

- $x^* \in A$  is called a maximal (Pareto Optimum) of  $A$  if  $\nexists x \in A : x > x^*$
- $x_* \in A$  is called a minimal (Pareto Optimum) of  $A$  if  $\nexists x \in A : x^* > x$

NB: On  $\mathbb{R}^1$ , if they exist, **maximal = maximum** and **minimal = minimum**

Eg.:

$$A = [0, 1] \in \mathbb{R}$$

### Lemma 56

The maximum  $A \subseteq \mathbb{R}^n$ , if it exists, is the unique maximal of  $A$ .

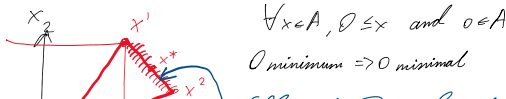
Proof:

Assume that  $\hat{x} \in A$  is a maximum of  $A$ , therefore  $\forall x \in A, x \leq \hat{x}$  and  $\hat{x}$  is also a maximal of  $A$ . Let  $x \in A$  with  $x \neq \hat{x}$ . Since (from the assumption)  $\hat{x}$  is a maximum of  $A$  we have  $\hat{x} \geq x$ , but  $x \neq \hat{x}$  therefore  $\hat{x} > x$ . We have  $x$  is not a maximal of  $A$ .

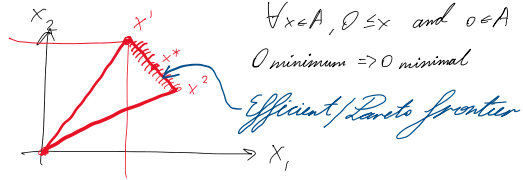
NB.

- The minimum of  $A \in \mathbb{R}^n$ , if it exists, is the unique minimal of  $A$ .
- $x^*$  maximum (minimum)  $\Rightarrow x^*$  maximal (minimal)
- $x^*$  maximal (minimal)  $\nRightarrow x^*$  maximum (minimum)

Eg.  
 $A \subseteq \mathbb{R}^2$



Eg.  
 $A \subseteq \mathbb{R}^2$



**Pareto (or Efficient) Frontier:** the set of all maximals (minimals) of  $A \in \mathbb{R}^n$

## Vector space

Consider a set  $A$  with two operations:

1. Addition:  $x, y \in A, x + y \in A$
2. Scalar multiplication:  $x \in A, a \in R, a * x \in A$

### Properties:

$(x, y, z \in A \text{ and } a, b \in R)$

1.  $x + y = y + x$
2.  $(x + y) + z = x + (y + z)$
3.  $x + 0 = 0 + x = x$
4.  $x + (-x) = -x + x = 0$
5.  $a(x + y) = ax + ay$
6.  $(a + b) * x = ax + bx$
7.  $1 * x = x * 1 = x$
8.  $a(bx) = (ab)x$

$A$  is called a vector space

Eg.  $R^n$  vector space;  $f: R \rightarrow R, R^R$  (set of all functions from  $R$  to  $R$ )

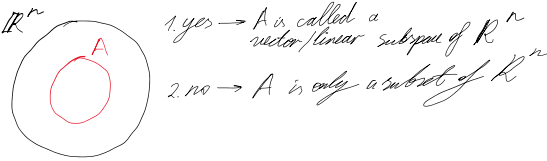
$R^R$  with elements functions;  $f, g \in R^R$

1.  $f + g \in R^R$
2.  $\forall x \in R, a * f \in R^R$   
+ the same 8 properties  
 $\forall x \in R, f(x) = 0$  - Neutral element for the addition.  
Therefore,  $R^R$  is a vector space  
 $R^N$  - all possible functions from  $N$  to  $R \rightarrow$  Sequences of real numbers

Eg.

Let  $R^n$  be a vector space and consider  $A \subseteq R^n$

Does  $A$  have the same structure of  $R^n$ ?



Definition.

Let  $V$  be a non-empty subset of  $R^n$ .  $V$  is called a vector subspace of  $R^n$  when:

1.  $\forall x, y \in V, x + y \in V$  ( $V$  is closed with respect to the addition)
2.  $\forall x \in V, \forall a \in R, ax \in V$  ( $V$  is closed with respect to the scalar multiplication)

Eg.

$V = \{x \in R^2: x_1 - 2x_2 = 0\}$ ,  $R^2$  is a vector space. Show that  $V$  is a linear space of  $R^2$

$V \neq \emptyset$  (for example  $x^* = (2, 1) \in V$ )

1. Suppose that  $x, y \in V$ . Then  $x + y \in R^2$  and  
 $(x_1 + y_1) - 2(x_2 + y_2) = x_1 - 2x_2 + y_1 - 2y_2 = 0$ , therefore  $V$  is closed w.r.t. the addition of vectors

2. Suppose that  $x \in V$  and  $a \in R$ . Therefore  $ax \in R^2$   
 $(ax_1) - 2(ax_2) = ax_1 - 2ax_2 = a(x_1 - 2x_2) = a * 0 = 0$   
Therefore,  $V$  is closed with respect to scalar multiplication.  
 $V$  is a linear subspace of  $R^2$

E.g.

$V = \{x \in R^2: x_1 + x_2 \leq 0\}$ . Is  $V$  a linear subspace of  $R^2$

$V \neq \emptyset$  ( $x^* = (-1, -1) \in V$ )

1. Suppose that  $x, y \in V$ , therefore  $x + y \in R^2$ . We have  
 $x_1 + x_2 = (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) \leq 0$ , and  $V$  is closed with respect to the addition

2. Assume that  $x \in V$  and  $a \in R$ . We have  $ax \in R^2$   
 $(ax_1) + (ax_2) = a(x_1 + x_2)$   
 $\exists a < 0: a(x_1 + x_2) > 0$ , therefore  $V$  is not closed w.r.t. the scalar multiplication  
Hence,  $V$  is not a linear subspace of  $R^2$

!!! Checks for both closings (w. f.t. + and -)

### Proposition 6.4

Let  $V$  be a non-empty subset of  $R^n$

$V$  is a vector subspace  $\Leftrightarrow ax + by \in V, \forall x, y \in V$  and  $a, b \in R$

NB.  $V$  vector subspace  $\Leftrightarrow \forall x, y \in V, \forall a, b \in R, ax + by \in V$  of  $x, y$  m/inf 2, 6

$\forall x, y \in V, \forall a, b \in R, ax + by \in V \Rightarrow V$  is a vector subspace

**Proof**

Assume that  $V$  is a vector subspace of  $R^n$ .

Consider  $x, y \in V$  and  $a, b \in R$ . From the premise, we have  $ax \in V$  and  $by \in V$ , and  $ax + by \in V$ . Therefore  $ax + by \in V$ .

Assume that  $ax + by \in V$  for any  $x, y \in V$  and  $a, b \in R$ .

Taking:

1.  $a = b = 1$  we have  $x + y \in V$  ( $V$  is closed w.r.t. the addition)
2.  $a = 1$  and  $b = 0$ , we have  $ax \in V$  ( $V$  is closed w.r.t. the scalar multiplication)

$V$  is linear subset of  $R^n$

E.g.

$V = \{x \in R^3: x_2 - x_3 = 0\}$ ,  $V \neq \emptyset, x^* = (2, 1, 1) \in V$

Consider  $x, y \in V$  and  $a, b \in R$ . Therefore,  $ax + by \in R^3$ . And we have

$$(ax_2 + by_2) - (ax_3 + by_3) = a(x_2 - x_3) - b(y_2 - y_3) = a * 0 + b * 0 = 0$$

Therefore  $ax + by \in V$ , and  $V$  is a linear subspace of  $R^3$

a.  $V$  is a linear subspace of  $R^n \Rightarrow 0 \in V$  (the neutral element of  $R^n$  must be an element of  $V$ )

b.  $0 \notin V \Rightarrow V$  is not a linear subspace of  $R^n$

Eg.

$V = \{x \in R^3: x_1 + x_2 - 2x_3 = 1\}$

$0 \in R^3$

$0 + 0 - 2 * 0 = 0 \neq 1, 0 \notin V$

Hence,  $V$  is not the vector space of  $R^3$

NB.

$V$  is linear subspace of  $R^n \Rightarrow 0 \in V$  (the neutral element of  $R^n$  must be an element of  $V$ )

$0 \notin V \Rightarrow V$  is NOT a linear subspace of  $R^n$

Eg.

$V = \{x \in R^2: x_1 + x_2 = 4\}$ . Is  $V$  a linear subspace of  $R^2$ ?

$0 \in V$ ?

$x_1 + x_2 = 0 + 0 = 0 \neq 4 \Rightarrow V$  is not a vector subspace of  $R^2$

Eg.

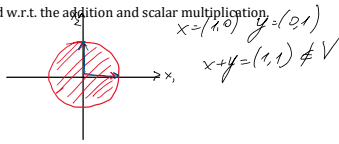
$V = \{x \in R^2: x_1^2 + x_2^2 \leq 4\}$ . Is  $V$  a linear subset of  $R^2$ ?

$$V \neq \emptyset (x = (1,1) \in V)$$

$$0 \in V?$$

$$x_1^2 + x_2^2 = 0 + 0 = 0 \leq 4 \Rightarrow 0 \in V$$

V is not closed w.r.t. the addition and scalar multiplication.



Eg.

Let  $a$  be a vector of  $R^n$ . Consider

$$V = \{x \in R^n : a \cdot x = 0 \text{ with } a \in R^n\}$$

We can show that  $V$  is a linear subspace of  $R^n (V \neq \emptyset)$

- Necessary condition:  $0 \in V?$

$$a \cdot x = a \cdot 0 = \sum_{i=1}^n a_i \cdot 0_i = n \cdot 0 = 0 \in V$$

- $x, y \in V$  and  $\alpha, \beta \in R$ . Therefore  $\alpha x + \beta y \in R^n$ , then

$$a(\alpha x + \beta y) = a \cdot (\alpha x) + a \cdot (\beta y) = \alpha(a \cdot x) + \beta(a \cdot y) = \alpha \cdot 0 + \beta \cdot 0 = 0$$

therefore  $V$  is a linear subspace of  $R^n$

Eg.

$$V = \{x \in R^4 : \underbrace{x_1 - x_2 + 2x_3 - x_4 = 0}_{a \cdot x = 0 \text{ given } a = (1, -1, 2, -1)}\}$$

### Proposition 68

Let  $V_1, V_2, \dots$  be a collection (finite or not finite) of linear subspaces of  $R^n$ . Then the intersection of  $V_1, V_2, \dots$  is a linear subspace of  $R^n$

Proof.

Assume that  $V_1, V_2, \dots$  are linear subspaces of  $R^n$ .

Therefore  $\forall i, V_i \neq \emptyset, \alpha x + \beta y \in V_i$  for any  $x, y \in V_i$  and  $\alpha, \beta \in R$ . We have to show that  $\cap_i V_i (\cap_{i=1}^n V_i \text{ or } \cap_{i=1}^{\infty} V_i)$  is also a linear subspace of  $R^n$ .

We have:

- $\cap_i V_i \neq \emptyset$  (because  $\forall i, V_i \neq \emptyset$ ) and  $0 \in \cap_i V_i$

Let  $x, y$  be elements of  $\cap_i V_i$  and  $\alpha, \beta \in R$

$$\alpha x + \beta y \in V_i, \alpha x_2 + \beta y_2 \in V_2$$

Therefore  $\alpha x + \beta y \in \cap_i V_i$  (It is closed w.r.t. the linear combination of  $x, y$  with coefficients  $\alpha, \beta$ ) and  $\cap_i V_i$  is a linear subspace of  $R^n$ .

NB. Proof by contradiction

Assume that  $V_1, V_2, \dots$  are subspaces of  $R^n (\forall x, y \in R^n, \forall \alpha, \beta \in R, \alpha x + \beta y \in V_i, \forall i)$

We suppose that the conclusion is false, i.e.  $\exists x, y \in R^n, \forall \alpha, \beta \in R, \alpha x + \beta y \notin \cap_i V_i$ . Then

$\alpha x + \beta y \in V_1$  or  $\alpha x + \beta y \notin V_2$  or ....

Therefore, at least a proposition exists such that  $\alpha x + \beta y \notin V_i$ , but this is impossible from the starting premise. Therefore,  $\cap_i V_i$  is a linear subspace of  $R^n$

E.g.  $a \cdot x = 0, a = (1, -1)$

$$V_1 = \{x \in R^3 : x_1 - x_2 = 0\} = \{a \cdot x = 0\}$$

$$V_2 = \{x \in R^3 : x_3 = 0\}$$

$V_1, V_2$  are both linear subspaces of  $R^3$

$$V_1 \cap V_2 = \{x \in R^3 : x_1 - x_2 = 0 \text{ and } x_3 = 0\}$$

$$V_1 \cap V_2 \neq \emptyset (\text{E.g. } x = (1, 1, 0) \in V_1 \cap V_2)$$

$0 \in V_1 \cap V_2?$

$x_1 - x_2 = 0, x_3 = 0$ , hence the answer is yes

Let  $x, y \in V_1 \cap V_2$  and  $\alpha, \beta \in R$ . Therefore  $\alpha x + \beta y \in R^3$

$$\begin{cases} (\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2) = 0 \\ \alpha x_3 + \beta y_3 = 0 \end{cases} \Rightarrow \begin{cases} \alpha(x_1 - x_2) + \beta(y_1 - y_2) = 0 \\ \alpha x_3 + \beta y_3 = 0 \end{cases} \Rightarrow \begin{cases} \alpha \cdot 0 + \beta \cdot 0 = 0 \\ \alpha \cdot 0 + \beta \cdot 0 = 0 \end{cases}$$

$\Rightarrow V_1 \cap V_2$  is a linear subspace of  $R^3$

NB.

Generic element or point of  $V_1 \cap V_2$

$$x = \begin{bmatrix} x_1 \\ x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

**Is  $V_1 \cup V_2$  a linear subset of  $R^3$ ?**

$$V_1 \cup V_2 = \{x \in R^3 : x_1 - x_2 = 0 \text{ or } x_3 = 0\}$$

$$\begin{cases} x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \in V_1 \cup V_2, y = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \in V_1 \cup V_2 \\ x + y = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \end{cases}$$

$$\begin{aligned} (x_1 + y_1) - (x_2 + y_2) &= 2 - 0 = 2 \neq 0 \\ x_3 + y_3 &= 2 + 0 \end{aligned}$$

$P_1$  and  $P_2$  are F

**$V_1 \cup V_2$  is not a linear subspace of  $R^3$**

NB.

If  $V_1, V_2$  are two linear subspaces of  $R^n$ . Then:

- $V_1 \cap V_2$  is a linear subspace of  $R^n$  (Prop)
- $V_1 \cup V_2$  in general is not a linear subspace of  $R^n$   
 $V_1, V_2$  subspaces  $\neq V_1 \cup V_2$  subspace of  $R^n$
- If  $V_2 \subseteq V_1$  then  $V_1 \cup V_2 = V_1$  and  $V_1 \cup V_2$  is a linear subspace of  $R^n$

Eg.

$$V_1 = \{x \in R^3 : x_3 = 0\} \text{ and } V_2 = \{x \in R^3 : x_1 = x_2 \text{ and } x = 0\}$$

$$V_2 \subseteq V_1 \text{ therefore } V_1 \cup V_2 = V_1 \text{ and } V_1 \cup V_2 \text{ is a linear subspace of } R^3$$

## Linear independence and dependence

N.B.

$$x^i = (x_1^i, x_2^i, \dots, x_n^i) \in R^n$$

$i$  - th vector of  $R^n$  with  $i = 1, \dots, m$

$$\text{Eg. } \left\{ \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\} = S, S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Def.  $\{x^1, \dots, x^m\}$  of  $R^n$ . This set is called a linearly independent set when

$$a_1 x^1 + a_2 x^2 + \dots + a_n x^n = 0 \Rightarrow a_1 = a_2 = \dots = a_n = 0$$

with  $a_1, a_2, \dots, a_n \in R$

NB.

$\{x^1, \dots, x^n\}$  or  $R^n$  is linearly dependent set if at least a non-zero coefficient  $a_i (i = 1, \dots, n)$  exists such that

$$a_1 x^1 + a_2 x^2 + \dots + a_n x^n = 0$$

$$\text{Eg. } \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\} = S$$

Is  $S$  linearly independent?

$$a_1 x^1 + a_2 x^2 = 0 \Rightarrow a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 - a_2 \\ 2a_1 + 3a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} a_1 - a_2 = 0 \\ 2a_1 + 3a_2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = a_2 \\ 5a_2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \end{cases}$$

Set  $S$  is linearly independent

Eg.  $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\} = S$ . Is  $S$  linearly independent?

NB.  $2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,  $a_1 = 2$  and  $a_2 = -1$ .  $S$  is linearly dependent

$$a_1 x^1 + a_2 x^2 = 0 \Rightarrow a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - a_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 + a_2 \\ 2a_1 + 4a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} a_1 + 2a_2 = 0 \\ 2a_1 + 4a_2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 + 2a_2 = 0 \\ a_1 + 2a_2 = 0 \end{cases} \Rightarrow a_1 + 2a_2 = 0 \Rightarrow a_2 = -\frac{a_1}{2}, a_1 \in \mathbb{R}$$

$$\left( a_1, -\frac{a_1}{2} \right), a_1 = 2, a_2 = -1$$

NB. The vector of  $\mathbb{R}^n$  given by  $e^1 = (1, 0, \dots, 0), e^2 = (0, 1, 0, \dots, 0), \dots, e^n = (0, 0, \dots, 1)$  are called the **unit vectors** or  $\mathbb{R}^n$

Eg.  $\{(1, 0), (0, 1)\}, \{(1, \mathbf{0}), (0, \mathbf{1}), (0, \mathbf{0}, 1)\}$

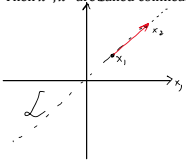
Set  $(e^1, e^2, \dots, e^n)$  is linearly independent  
 $a_1 e^1 + a_2 e^2 + \dots + a_n e^n = 0$   
 $\Rightarrow a_1(1, 0, \dots, 0) + a_2(0, 1, 0, \dots, 0) + \dots + a_n(0, 0, \dots, 1) = (0, \dots, 0)$   
 $\Rightarrow (a_1, a_2, \dots, a_n) = (0, 0, \dots, 0) \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ \dots \\ a_n = 0 \end{cases}$

NB. If in  $S = \{x^1, \dots, x^n\}$  of  $\mathbb{R}^n$  we have 0 then  $S$  is linearly dependent

Eg.  $\{(1, 2, 3), (-1, 0, 1), (0, 0, 0)\}$   
 $a_1 x^1 + a_2 x^2 + a_3 x^3 = 0$   
 $0x^1 + 0x^2 + 2x^3 = 0$

At least a coefficient  $a_i (i = 1, 2, 3) \neq 0$  exists then  $S$  is linearly dependent

NB.  $\{x^1, x^2\}$  linearly dependent set  
 Then  $x^1, x^2$  are called collinear vectors



$x^1$  and  $x^2$  are **collinear** if

- $x^1 = 0$
- $x^2 = 0$
- $\exists \alpha \neq 0: x^1 = \alpha x^2$

Eg.  $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$   
 $x^1 = \frac{1}{2} x^2$  or  $x^2 = 2x^1$

Eg.  $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$   
 Is  $S$  linearly independent?  
 $a_1(1, 0, 0) + a_2(1, 1, 0) + a_3(1, 1, 1) = (0, 0, 0)$   
 $(a_1 + a_2 + a_3, a_2 + a_3, a_3) = (0, 0, 0)$   
 $\begin{cases} a_1 + a_2 + a_3 = 0 \\ a_2 + a_3 = 0 \\ a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_1 \\ a_2 \\ a_3 \end{cases} \Rightarrow S \text{ is linearly independent}$   
 $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\} \rightarrow \begin{cases} s_1 = \{(1, 0, 0), (1, 1, 1)\} \\ s_2 = \{(1, 1, 0), (1, 1, 1)\} \end{cases}$

### Proposition 76

Subsets of a linearly independent sets are linearly dependent

Eg.  $x^1, x^2$   
 $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$   
 $a_1 x^1 + a_2 x^2 = x$   
 $x$  is a linear combination of  $x^1, x^2$  with coefficient  $a_1, a_2$   
 $a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 + a_2 \\ 2a_1 - a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} a_1 + a_2 = 3 \\ 2a_1 - a_2 = 2 \end{cases} \Rightarrow \begin{cases} a_1 = 3 - a_2 \\ 6 - 2a_2 - a_2 = 2 \end{cases}$   
 $\Rightarrow \begin{cases} a_1 = \frac{5}{3} \\ a_2 = \frac{4}{3} \end{cases}$

Definition:

$x \in \mathbb{R}^n$ .  $x$  is said to be a linear combination of  $\{x^1, \dots, x^n\}$  of  $\mathbb{R}^n$  if a set of scalars  $\{a_1, \dots, a_n\}$  exists such that  $x = a_1 x^1 + a_2 x^2 + \dots + a_n x^n$

Given a set  $S = \{x^1, x^2, \dots, x^m\}$  of  $\mathbb{R}^n$ ,  $x \in \mathbb{R}^n$  is said to be a linear combination of  $x^1, x^2, \dots, x^m$  if a set of  $m$  scalars  $\{a_1, \dots, a_m\}$  exists such that

$$x = a_1 x^1 + a_2 x^2 + \dots + a_m x^m = \sum_{s=1}^m a_s x^s$$

Eg.  $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}, x = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \in \mathbb{R}^2, n_1, n_2 \in \mathbb{R}$   
 $x = a_1 x^1 + a_2 x^2 \Rightarrow \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} a_1 - a_2 \\ a_2 \end{bmatrix} \Rightarrow \begin{cases} a_1 - a_2 = n_1 \\ a_2 = n_2 \end{cases} \Rightarrow \begin{cases} a_1 = n_1 + n_2 \\ a_2 = n_2 \end{cases}$   
 Therefore a set of scalars exists, therefore  $x$  is a linear combination of  $x^1, x^2$  with coefficients  $a_1 = n_1 + n_2$ ,  $a_2 = n_2$   
 $x = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, a_1 = 6, a_2 = 2, x = 6 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$a_1 \neq a_2$  linear system imp.  $x$  is not a linear combination of  $x^1, x^2$

Eg.  $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}, x^1, x^2$  are collinear vectors  $x = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2, a_1, a_2 \in \mathbb{R}$   
 $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = d_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} d_1 + 2d_2 \\ d_1 + 2d_2 \end{bmatrix} \Rightarrow \begin{cases} d_1 + 2d_2 = a_1 \\ d_1 + 2d_2 = a_2 \end{cases}$   
 $a_1 = a_2 \Rightarrow d_1 + 2d_2 = a_1 \Rightarrow d_1 = a_1 - 2d_2, d_2 \in \mathbb{R}$

$(a_1 - 2d_2, d_2)$  Set of solution of the linear system with  $a_1 = a_2$ ,  $x$  can be written as a linear combination of  $x^1, x^2$

$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  is not a linear combination of  $x^1, x^2$

$x^1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$  is a linear combination of  $x^1, x^2: d_1 = 3 - 2d_2, d_2 \in \mathbb{R}$

### Theorem 79

$S$  is a finite set of vectors of  $\mathbb{R}^n$  with at least two elements.

$S$  is linearly dependent  $\Leftrightarrow$  at least an element of  $S$  exists such that it can be written as a linear combination of the other elements of  $S$

**Proof**  
 (⇒)  
 Assume that  $S = \{x^i\}_{i=1}^m$  is a linearly dependent set of  $R^n$ , that is at least a scalar  $d_i \neq 0 (i = 1, \dots, m)$  exists such that  
 $d_1 x^1 + d_2 x^2 + \dots + d_m x^m = 0$   
 We suppose that  $d_1 \neq 0$ . We have  
 $d_1 x^1 = -d_2 x^2 - \dots - d_m x^m \Rightarrow x^1 = -\frac{d_2}{d_1} x^2 - \frac{d_3}{d_1} x^3 - \dots - \frac{d_m}{d_1} x^m$   
 Therefore  $x^1$  is a linear combination of the other elements of  $S$   
 (⇐)  
 Assume that a vector  $x^i$  of  $S = \{x^i\}_{i=1}^m$  is a linear combination of the other elements. We suppose that  $x^1$  can be written as a linear combination.  
 We have:  

$$\begin{cases} x^1 = d_2 x^2 + \dots + d_m x^m \\ \exists \text{ a set of scalars } \{d_2, \dots, d_m\} \end{cases} \Rightarrow -x^1 + d_2 x^2 + \dots + d_m x^m = 0$$
  
 Indicating with  $b_1 = -1$  and  $b_i = d_i \forall i \geq 2$   
 We have  

$$b_1 x^1 + b_2 x^2 + \dots + b_m x^m = 0 \Rightarrow \sum_{i=1}^m b_i x^i = 0$$
 and  $\{x^i\}_{i=1}^m$  is a linearly dependent set of  $R^n$

Eg.  $S = \{(1,2,0), (-1,1,1), (0,3,1)\}$   $x^3 = x^1 + x^2$   
 $S$  is linearly dependent  
**Corollary**  
 Let  $S \neq \{0\}$  be a finite set of  $R^n$ . Set  $S$  is linearly independent ⇔ none of vectors of  $S$  is a linear combination of the other vectors of  $S$ .

Eg.  $S = \{(1,0,0), (1,1,0), (1,1,1)\}$   
 $S$  is linearly independent  

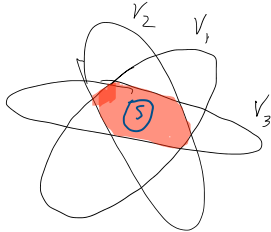
$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = d_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d_1 + d_2 \\ d_1 + d_2 \\ d_2 \end{bmatrix} \Rightarrow \begin{cases} d_1 + d_2 = 1 \\ d_1 + d_2 = 0 \\ d_2 = 0 \end{cases}$$
  

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} d_1 + d_2 + d_3 = 0 \\ d_2 + d_3 = 0 \\ d_3 = 0 \end{cases} \Rightarrow \begin{cases} d_1 = 0 \\ d_2 = 0 \\ d_3 = 0 \end{cases}$$
  
*can't exist*

### Generated Subspaces

- $S$  is a non-empty set of vectors of  $R^n$
- $\{V_i\}$  is a collection (finite and not finite) of vector subspaces, every subspace of which contains  $S (S \subseteq V_i \forall i)$

$\cap_i V_i$  is a vector subspace of  $R^n$  which contains  $S$   
 $\cap_i V_i$  is the smallest vector subspace of  $R^n$  with  $S$



$\cap_i V_i$  is called the vector subspace of  $R^n$  **generated** or spanned by  $S$ : **span  $S = \cap_i V_i$**

NB.  
**Span  $S$**  can be represented using the linear combination of  $S$

**Theorem 83.**  
 Let  $S$  be a set of vectors of  $R^n$  and let  $x$  be a vector of  $R^n$   
 $x \in \text{span } S \Leftrightarrow x$  is a linear combination of vectors of  $S$   
**Proof**  
 (⇒)  
 Assume that  $x$  is a linear combination of a finite set  $\{x^i\}_{i=1}^m$  of vectors.  
 A set  $\{d_i\}_{i=1}^m$  exists such that  

$$x = d_1 x^1 + d_2 x^2 + \dots + d_m x^m = \sum_{i=1}^m d_i x^i$$
  
 Since span is a vector subspace of  $R^n$ , we have  
 $d_1 x^1 + d_2 x^2 \in \text{span } S \Rightarrow (d_1 x^1 + d_2 x^2) + d_3 x^3 \in \text{span } S \Rightarrow (d_1 x^1 + d_2 x^2 + \dots + d_{m-1} x^{m-1}) + d_m x^m \in \text{span } S \Rightarrow x \in \text{span } S$   
 (⇐)  
 Let  $V$  be the set of all the vectors of  $R^n$  such that can be written as a linear combination of vectors  $S$ .  
 $V = \{x \in R^n: x = d_1 x^1 + d_2 x^2 + \dots + d_m x^m, d_1, d_2, \dots, d_m \in R\}$   
 $V$  is a vector of subspace of  $R^n$  ( $V$  is closed w.r.t. the addition and scalar multiplication operations)  
 1.  $x, y \in V$   
 $x + y = (d_1 x^1 + \dots + d_m x^m) + (b_1 x^1 + \dots + b_m x^m)$   
 $= (d_1 + b_1)x^1 + (d_2 + b_2)x^2 + \dots + (d_m + b_m)x^m = g_1 x^1 + g_2 x^2 + \dots + g_m x^m$   
 $g_1, g_2, \dots, g_m \in R$   
 Therefore,  $x + y \in V$  ( $V$  is closed w.r.t. the addition)  
 2.  $x \in V, w \in R$ . Is  $wx \in V$ ?  
 $wx = w[d_1 x^1 + \dots + d_m x^m] = (wd_1)x^1 + \dots + (wd_m)x^m = t_1 x^1 + \dots + t_m x^m, t_1, \dots, t_m \in R$   
 therefore  $wx \in V$  ( $V$  is closed w.r.t. scalar multiplication).  $V$  is a vector subspace of  $R^n$  and  $S \subseteq V$ . Therefore  $S \subseteq \text{span } S \subseteq V$  and  $x \in \text{span } S$  can be written as a linear combination of vectors of  $S$ .

Eg.  $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \subseteq R^2$ . Find the span of  $S$ .  
 Let  $x$  be an element of  $R^2$   
 $x = d \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} d+b \\ b \end{bmatrix} \in R^2 \Rightarrow \text{span } S = R^2$   
*starting from S we can generate R^2*

Eg.  $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$   
 Find the span of  $S = R^2$   
 $x = d \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} d+2b \\ d+b \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$   
**Span  $S = R \subseteq R^2$**   
*starting from S we can generate R*

**Corollary**  
 Let  $S$  be a set of vectors of  $R^n$ . If  $x \in R$  is a linear combination of vectors of  $S$  then **span  $S = \text{span } (S \cup \{x\})$**   
 Eg.

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}. \text{ Find the span } S \in \mathbb{R}^2$$

$$x = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 2 \end{bmatrix} + c \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} a + 2b + 3c \\ a + 2b + 3c \end{bmatrix} = \begin{bmatrix} d \\ d \end{bmatrix} \Rightarrow \text{span } S = R$$

### Bases of a vector space

Eg.  
 $S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$  and  $S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$   
 $S_1$  is linearly indep.  $S_2$  is linearly dep.  
 $S_1 \in \mathbb{R}^2$  and  $S_2 \in \mathbb{R}^2$   
 $x \in \mathbb{R}^2, x = \begin{bmatrix} a \\ b \end{bmatrix}, a, b \in \mathbb{R}$

$$\begin{bmatrix} a \\ b \end{bmatrix} = d \begin{bmatrix} 1 \\ 0 \end{bmatrix} + p \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} d + p = a \\ p = b \end{cases} \Rightarrow \begin{cases} d = a - p \\ p = b \end{cases} - \text{Unique solution}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} d = 2 - 1, p = 1$$

$$x = \begin{bmatrix} a \\ b \end{bmatrix} = d \begin{bmatrix} 1 \\ 0 \end{bmatrix} + p \begin{bmatrix} 1 \\ 1 \end{bmatrix} + g \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} d + p + 2g = a \\ p + g = b \end{cases} \Rightarrow \begin{cases} d = a - 2g - b + g \\ p = b - g \\ g \in \mathbb{R} \end{cases}$$

*3 variables  
2 equations*

$$(a - b - g, b - g, g)_{g \in \mathbb{R}}$$

### Bases of a linear space

Definition: Let  $S$  be a finite set of vectors of  $\mathbb{R}^n$ . If

1.  $\text{span } S = \mathbb{R}^n$
2.  $S$  is a linearly independent set of vectors

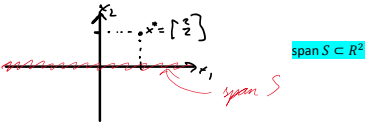
Then  $S$  is said to be a bases of  $\mathbb{R}^n$

Eg.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\} \in \mathbb{R}^2. \text{ Is } S \text{ a bases of } \mathbb{R}^2?$$

1. Let  $x$  be a vector of  $\mathbb{R}^2$ . Therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = d_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 + 2d_2 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = d_1 + 2d_2 \\ x_2 = 0 \end{cases}$$



$S$  is not a bases of  $\mathbb{R}^2$

Eg2.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \in \mathbb{R}^2$$

Is  $S$  a bases of  $\mathbb{R}^2$ ?

1.  $\text{Span } S = \mathbb{R}^2$ ? (we can find any  $x \in \mathbb{R}^2$  using vectors of  $S$ )

Let  $x$  be a point/vector/element of  $\mathbb{R}^2$ . We have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = d_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + d_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} d_1 + d_3 = x_1 \\ d_2 + d_3 = x_2 \end{cases}$$

*variables 3 > 2 equations  
hence \infty solutions*

$$\Rightarrow \begin{cases} d_1 = x_1 - d_3 \\ d_2 = x_2 - d_3, (x_1 - d_3, x_2 - d_3, d_3), d_3 \in \mathbb{R} \\ d_3 = \mathbb{R} \end{cases}$$

2. Is  $S$  linearly dependent?

$$d_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + d_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} d_1 + d_3 = 0 \\ d_2 + d_3 = 0 \end{cases} \Rightarrow \infty \text{ solutions}$$

$\rightarrow S$  is not a linearly independent set of  $\mathbb{R}^2$

$S$  is not a bases of  $\mathbb{R}^2$

Eg3.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \in \mathbb{R}^2. \text{ Is } S \text{ a bases of } \mathbb{R}^2? \text{ (using the definition)}$$

1. Let  $x$  be a vector of  $\mathbb{R}^2$ .  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = d_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} d_1 + d_2 = x_1 \\ d_2 = x_2 \end{cases}, (x_1 - x_2, x_2) \Rightarrow$   
 $\Rightarrow \text{span } R = S \rightarrow$  we can find any point of  $\mathbb{R}^2$  using vectors of  $S$ .

2.  $0 \in \mathbb{R}^2, \begin{bmatrix} 0 \\ 0 \end{bmatrix} = d_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} d_1 + d_2 = 0 \\ d_2 = 0 \end{cases} \Rightarrow \begin{cases} d_1 = 0 \\ d_2 = 0 \end{cases}$

$S$  is linearly independent and  $S$  is a bases of  $\mathbb{R}^2$

### Theorem 89

Let  $S$  be a finite set of  $\mathbb{R}^n$ .  $S$  is a bases of  $\mathbb{R}^n \Leftrightarrow x \in \mathbb{R}^n$  can be written in a unique way as a linear combination of vectors of  $S$ .

Proof.

$S$  bases of  $\mathbb{R}^n \Rightarrow \mathbb{R}^n$  can be written in a unique way  $x = \sum_{i=1}^m d_i x^i$

Let  $S = \{x^i\}_{i=1}^m$  be a bases of  $\mathbb{R}^n$ . Therefore any  $x \in \mathbb{R}^n$  can be written as a linear combination of vectors of  $S$  and  $S$  is linearly independent.

Given  $x \in \mathbb{R}^n$ , we suppose that  $x$  can be written in two different ways: two sets of scalars  $\{a_i\}_{i=1}^m$  and  $\{b_i\}_{i=1}^m$  exists such that

$$x = \sum_{i=1}^m a_i x^i \text{ and } x = \sum_{i=1}^m b_i x^i$$

We have

$$\sum_{i=1}^m a_i x^i = \sum_{i=1}^m b_i x^i \Rightarrow \sum_{i=1}^m a_i x^i - \sum_{i=1}^m b_i x^i = 0 \Rightarrow \sum_{i=1}^m [a_i x^i - b_i x^i] = 0 \Rightarrow \sum_{i=1}^m (a_i - b_i) \cdot x^i = 0$$

$\Rightarrow 0 \Rightarrow \forall i, a_i - b_i = 0 \Rightarrow \forall i, a_i = b_i$   
 So  $x \in \mathbb{R}^n$  can be written in a unique way as a linear combination of vectors of  $S$

Eg.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\} \in \mathbb{R}^3$$

Is  $S$  a bases of  $\mathbb{R}^3$ ?

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = d_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + d_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} d_1 + d_2 + d_3 = x_1 \\ 2d_2 + d_3 = x_2 \\ -d_3 = x_3 \end{cases}$$

*unique solution  
2 \infty solutions*

Linear system  $\begin{cases} * \text{ Possible (at least one solution exists)} \\ * \text{ Impossible (no solution exists)} \end{cases}$

$$\begin{cases} d_1 + d_2 - x_3 = x_1 \\ 2d_2 - x_3 = x_2 \\ d_3 = -x_3 \end{cases} \Rightarrow \begin{cases} d_1 = x_1 + x_3 - \frac{x_3 + x_2}{2} \\ d_2 = \frac{x_3 + x_2}{2} \\ d_3 = -x_3 \end{cases} \Rightarrow \begin{cases} d_1 = \frac{2x_1 + x_3 - x_2}{2} \\ d_2 = \frac{x_2 + x_3}{2} \\ d_3 = -x_3 \end{cases}$$

*unique solution*

We have a unique solution, so  $S$  is a basis of  $\mathbb{R}^3$

NB.  
 $\{e^1, e^2, \dots, e^n\}$  is called the standard bases of  $R^n$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

Standard bases of  $R^2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$$\text{Standard bases of } R^3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \begin{matrix} e^1 \\ e^2 \\ e^3 \end{matrix}$$

Given  $x \in R^n$ , we have

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$= x_1 e^1 + x_2 e^2 + \dots + x_n e^n = \sum_{i=1}^n x_i e^i$$

$x$  can be written in a unique way as a linear combination of  $\{e^1, e^2, \dots, e^n\}$  and the coefficients of the components of  $x$ .

Eg.

$$x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = 2e^1 - e^2 + 3e^3$$

$$x = e^1 + 2e^2 - 3e^3 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

### Theorem

Let  $\{x^1, \dots, x^k\}$  be a linearly independent set of  $R^n$  with  $k \leq n$ . Then  $n - k$  vectors  $\{x^{k+1}, \dots, x^n\}$  exist such that  $\{x^1, x^2, \dots, x^n\}$  is a bases of  $R^n$

Eg.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, k = 2 \leq 3 = n, n - k = 1$$

$$S' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

### Corollary

1. Every linearly independent set of  $R^n$  with  $n$  elements is a basis of  $R^n$ .
2. Every linearly independent set of  $R^n$  has at most  $n$  elements.

Eg.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, S \text{ is not a bases}$$

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\} S \text{ is linearly independent} \Rightarrow S \text{ is a bases of } R^2$$

### Theorem

All the bases of  $R^n$  have the same number of vectors/elements

Eg.

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}; S' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}; S'' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

dimension (dim)  $R^n = n$ . the dimension of  $R^n$  is the number of vectors of bases of  $R^n$

We can extend all the theorems for a vector/ linear subspace  $V \in R^n$

Eg.

$$V_1 = \{x \in R^3: x_1 - x_2 + x_3 = 0\} \text{ and } V_2 = \{x \in R^3: x_1 = x_2 \wedge x_3 = 0\}$$

1.  $V_1$  is a linear subspace of  $R^3$

$$x = \begin{bmatrix} x_1 \\ x_1 + x_3 \\ x_3 \end{bmatrix} \in V_1 \quad x = \begin{bmatrix} x_1 \\ x_1 + x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$B_{V_1} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \dim(V_1) = 2$$

2.  $V_2$  is a linear subspace of  $R^3$

$$x = \begin{bmatrix} x_1 \\ x_1 \\ 0 \end{bmatrix} \in V_2; x = \begin{bmatrix} x_1 \\ x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, B_{V_2} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\dim(V_2) = 1 - \text{No}$$

Type equation here.

## Euclidean Structure

NB.

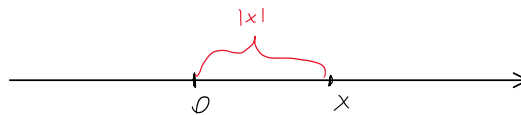
- Given  $x \in R^n, x * x = \sum_{i=1}^n x_i x_i = \sum_{i=1}^n x_i^2 \geq 0, \forall i, x_i^2 \geq 0$
- $x * x = 0 \Leftrightarrow x = 0$   
 $\begin{cases} x * x = 0 \Rightarrow x = 0 \text{ and } x = 0 \Rightarrow x * x = 0 \\ x \neq 0 \Rightarrow x * x \neq 0 (x * x > 0) \end{cases}$

Absolute value:

Definition:

$x \in R$ . The positive number ( $|x|$ ) defined by  $|x| \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$  is called the absolute value of  $x$

Geometrical interpretation:



## Properties:

- $|x| \geq 0, |x| = 0 \Leftrightarrow x = 0$
- $|x * y| = |x| |y|$  and  $\left| \frac{x}{y} \right| = \frac{|x|}{|y|}$  with  $y \neq 0$
- $|x + y| \geq |x| + |y|$  - Triangle inequality  
and the inequality sign holds if and only if  $x, y$  have the same sign

NB.

$$|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|$$

$$\forall x, y \in R, |x - y| \leq |x| + |y|$$

Proof of the triangle inequality. Let  $x, y \in R$ , we have

$$|x + y|^2 = (x + y)^2 = x^2 + 2xy + y^2 = |x|^2 + 2xy + |y|^2 \leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$$

Taking the square root of each side we obtain

$$\sqrt{|x + y|^2} \leq \sqrt{(|x| + |y|)^2} \Rightarrow |x + y| \leq |x| + |y|$$

NB.

$$1. \forall x, y \in R, x * y \leq |x| |y|$$

$$2. \forall x \in R, x \leq |x| \text{ and } -x \leq |x| \left( \begin{array}{l} x \leq |x| \text{ and } y \leq |y| \\ x + y \leq |x| + |y| \end{array} \right)$$

Let  $x, y$  be real numbers. We have

$$\begin{cases} x + y \leq |x| + |y| \\ -x - y \leq |x| + |y| \end{cases} \Rightarrow \begin{cases} x + y \leq |x| + |y| \\ x + y \geq -|x| - |y| \end{cases} \Rightarrow -|x| - |y| \leq x + y \leq |x| + |y| \Rightarrow \\ \Rightarrow -(|x| + |y|) \leq x + y \leq |x| + |y| \Rightarrow |x + y| \leq |x| + |y|$$

NB.

Let  $x$  be a real number. We have  $\max(x, -x) \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} = |x|$

$$\forall x \in R, \max(x, -x) = |x|$$

Eg.

$$|x| * \max(x, -x) \leq 0 \Rightarrow |x|^2 \leq 0 \Rightarrow x = 0$$

$$\min(x, -x) = \begin{cases} x & \text{if } x < 0 \\ -x & \text{if } x \geq 0 \end{cases} = -|x|$$

$$\forall x \in \mathbb{R}, \min(x, -x) = -|x|$$

Eg.

Let  $x \in \mathbb{R}$ . Solve  $\max(x, -x) + x * \min(x, -x) \leq 0$

$$I \Rightarrow |x| - x|x| \leq 0 \Rightarrow |x|(1 - x) \leq 0$$

$$\begin{cases} |x| \geq 0 \\ 1 - x \leq 0 \end{cases} \text{ or } \begin{cases} |x| \leq 0 \\ 1 - x \geq 0 \end{cases}$$

$$\begin{cases} \forall x \in \mathbb{R} \\ x \geq 1 \end{cases} \text{ or } \begin{cases} x = 0 \\ x \leq 1 \end{cases} \Rightarrow x \geq 1 \text{ or } x = 0$$

$$S = \{0\} \cup [1, +\infty)$$

$$\inf S = 0 \in S \Rightarrow \min S = 0$$

$$\nexists \sup S, \nexists \max S$$

$$S_* = (-\infty, 0]; S^* = \emptyset$$

## Lemma 106

Let  $x, y \in \mathbb{R}$ . Then  $||x| - |y|| \leq |x - y|$

*Proof.*

Let  $x, y$  be real numbers. Then

$$||x| - |y||^2 = (|x| - |y|)^2 = |x|^2 - 2|x||y| + |y|^2 = x^2 - 2|x||y| + y^2 \leq x^2 - 2xy + y^2 = (x - y)^2$$

Taking the square root of each side we get:

$$||x| - |y|| \leq |x - y|$$

**NB**

$$\forall x, y \in \mathbb{R}, xy \leq |x||y| \text{ and } -|x||y| \leq -xy$$

**NB.**

$$\forall x, y \in \mathbb{R} \quad |x| = |(x - y) + y| \leq |x - y| + |y|$$

$$|x| - |y| \leq |x - y|$$

By symmetry we have  $|y| - |x| \leq |x - y|$

$$\begin{cases} |x| - |y| \leq |x - y| \\ |y| - |x| \leq |x - y| \end{cases} \Rightarrow \begin{cases} |x| - |y| \leq |xy| \\ -( |x| - |y| ) \leq |xy| \end{cases} \Rightarrow \max(|x| - |y|, -( |x| - |y| )) \leq |xy| \Rightarrow ||x| - |y|| \leq |x - y|$$

$$\max(x, -x) = |x|$$

## Norm of a vector of $\mathbb{R}^n$

We generalize the notion of  $|x|$  in  $\mathbb{R}$  in another notion in  $\mathbb{R}^n$

**Norm: definition**

$x \in \mathbb{R}^n$ . The positive real number ( $\|x\|$ ) defined by  $\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\sum_{i=1}^n x_i^2}$  is called the norm of  $x \in \mathbb{R}^n$

Eg.

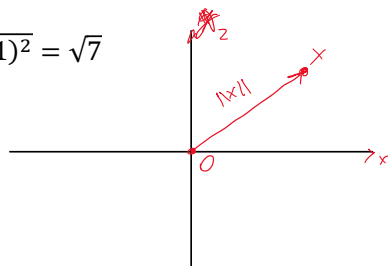
$$x = (-1, 1, 2, -1) \in \mathbb{R}^4, \|x\| = \sqrt{(-1)^2 + 1^2 + 2^2 + (-1)^2} = \sqrt{7}$$

**NB**

$$1. \quad n = 1: \|x\| = \sqrt{x_1^2} = |x|$$

$$2. \quad n = 2: \|x\| = \sqrt{x_1^2 + x_2^2}$$

Geo interpretation:  $\|x\|$  is the distance of  $x$  from 0



Properties:

$x, y \in R^n$  and  $a \in R$

1.  $\|x\| \geq 0$
2.  $\|x\| = 0 \Leftrightarrow x = 0$
3.  $\|a * x\| = |a| * \|x\|$

NB.

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{x * x}$$

NB.

- i.  $\|x\|^2 = (\sqrt{x * x})^2 = x * x \geq 0 = 0^2 \Rightarrow \|x\|^2 \geq 0^2 \Rightarrow \|x\| \geq 0$
- ii.  $\|a * x\| = \sqrt{(ax)(ax)} = \sqrt{a^2(x * x)} = |a|\sqrt{x * x} = |a|\|x\|$

## Theorem (Cauchy-Schwartz)

Let  $x, y \in R^n$ . Then

$$|x * y| \leq \|x\| * \|y\|$$

and the equality sign holds if and only if  $x$  and  $y$  are collinear vectors:

Eg.

$$x = (-1, 2), y = (3, 1)$$

$$x * y = (-1) * 3 + 2 * 1 = -1, \quad |x * y| = |-1| = 1$$

$$\|x\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

$$\|y\| = \sqrt{3^2 + 1^2} = \sqrt{10}$$

$$\|x\|\|y\| = \sqrt{5} * \sqrt{10} = \sqrt{50} = 5\sqrt{2}$$

$$|x * y| = 1 \leq 5\sqrt{2} = \|x\|\|y\|$$

### Corollary 110

Let  $x, y \in R^n$ . We have

$$\|x + y\| \leq \|x\| + \|y\|$$

Proof

Let  $x, y \in R^n$ . We have

$$\|x + y\|^2 = \left( \sqrt{(x + y)(x + y)} \right)^2 = (x + y)(x + y) = x * (x + y) + y(x + y) = x * x + x * y + y * x + y * y =$$

$$= x * x + 2x * y + y * y \leq \|x\|^2 + 2|xy| + 2|y|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

Taking the square root of each side

$$\|x + y\| \leq \|x\| + \|y\|$$

### Proposition 111.

Let  $x, y \in R^n$ . Then

$$\left| \|x\| - \|y\| \right| \leq \|x - y\|$$

Proof.

$x, y \in R^n$ . We observe that

$$\bullet \forall x \in R^n, \|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\| \Rightarrow \|x\| - \|y\| \leq \|x - y\|$$

$$\bullet \forall y \in R^n, \|y\| = \dots \text{By symmetry we have } \|y\| - \|x\| \leq \|x - y\|$$

$$\begin{cases} \|x\| - \|y\| \leq \|x - y\| \\ \|y\| - \|x\| \leq \|x - y\| \end{cases} \Rightarrow \begin{cases} \|x\| - \|y\| \leq \|x - y\| \\ -(\|x\| - \|y\|) \leq \|x - y\| \end{cases} \Rightarrow \max(\|x\| - \|y\|, -(\|x\| - \|y\|)) \leq \|x - y\|$$

$$\Rightarrow \left| \|x\| - \|y\| \right| \leq \|x - y\|$$



NB.

Any vector  $x \in R^n$  with  $\|x\| = 1$  is called normalized vector

## Orthogonality

Eg.

$$e^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \|e^1\| = 1$$

$x, y \in R^n$ . Vectors  $x$  and  $y$  are called orthogonal vectors ( $x \perp y$ ) if  $x \cdot y = 0$

Eg.

$$x = (1, -1), y = (-1, -1).$$

$$x \cdot y = 1 \cdot (-1) + (-1) \cdot (-1) = -1 + 1 = 0$$

$x, y$  are orthogonal vectors

NB.

$\{e^1, e^2, \dots, e^n\}$  – Set of standard vectors of  $R^n$

$$e^i \cdot e^j = 0 \quad \forall i, j = 1, \dots, n \text{ with } i \neq j$$

## Theorem (Pythagoras)

$x, y \in R^n$ .

$$\text{If } x \perp y \text{ then } \|x + y\|^2 = \|x\|^2 + \|y\|^2$$

Proof:

Let  $x, y$  be orthogonal vectors, i.e.  $x \cdot y = 0$ .

We have:

$$\begin{aligned} \|x + y\|^2 &= (x + y) \cdot (x + y) = x \cdot (x + y) + y \cdot (x + y) = x \cdot x + x \cdot y + x \cdot y + y \cdot y \\ &= \|x\|^2 + \underbrace{x \cdot y}_{=0} + \underbrace{x \cdot y}_{=0} + \|y\|^2 = \|x\|^2 + \|y\|^2 \end{aligned}$$

### Definition

A set of vectors  $R^n$  is said to be orthogonal if its elements are pairwise orthogonal vectors.

Eg.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \rightarrow \begin{cases} e^1 \cdot e^2 = 0 \\ e^2 \cdot e^3 = 0 \\ e^1 \cdot e^3 = 0 \end{cases} \rightarrow A \text{ is a set of orthogonal vectors}$$

## Proposition

Any orthogonal set of vectors that doesn't contain the zero vector ( $0 \in R^n$ ) is linearly independent.

NB. Set of **orthonormal** vectors of  $R^n$  if:

1. Set of orthogonal vectors
2. Each vector has norm with value 1

Eg.

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

$$\|x\| = \sqrt{2}$$

$$\|y\| = \sqrt{2}$$

$$S^* = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}, \|x\| = \|y\| = 1$$

Hence,  $S^*$  is a set of orthonormal vectors

### Proposition

Let  $\{x^1, x^2, \dots, x^k\}$  be an orthogonal set of vectors of  $R^n$ . We have:

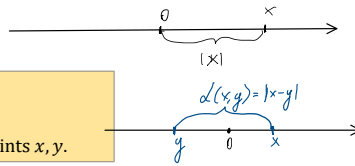
$$\left\| \sum_{i=1}^k x^i \right\|^2 = \sum_{i=1}^k \|x^i\|^2$$

# Chapter 5 - Topological structure

martedì 19 settembre 2023 08:32



NB  
 $x \in \mathbb{R}, |x|$  is the distance between  $x$  and  $0$  on a real line



**Definition**  
 $x, y \in \mathbb{R}$ . The positive real number  $d(x, y)$  defined by  $d(x, y) = |x - y|$  is called the **distance** between the two points  $x, y$ .

Eg.  
 $x = 2, y = 7, d(x, y) = |2 - 7| = |-5| = 5$

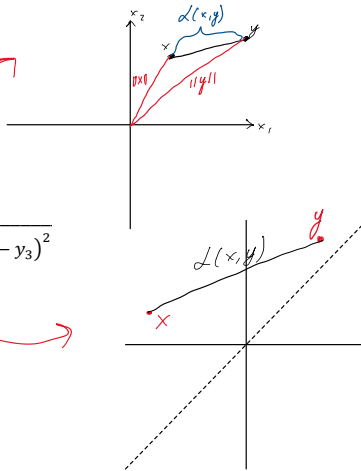
**Definition**  
 $x, y \in \mathbb{R}^n$ . The positive real number  $d(x, y)$  defined by  $d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = \sqrt{(x - y) * (x - y)}$  is called the **distance** between  $x$  and  $y$  on  $\mathbb{R}^n$

Eg.  
 $x = (1, -1, 0, 4), y = (0, 1, 2, -1)$   
 $x - y = (1, -1, 0, 4) - (0, 1, 2, -1) = (1, -2, -2, 5)$   
 $d(x, y) = \sqrt{|x - y|} = \sqrt{1^2 + (-2)^2 + (-2)^2 + 5^2} = \sqrt{34}$

$$d(x, y) = \sqrt{\sum_{i=1}^4 (x_i + y_i)^2} = \sqrt{(1 - 0)^2 + (-1 - 1)^2 + (0 - 2)^2 + (4 - (-1))^2} = \sqrt{34}$$

**Remarks:**

- $n = 1, d(x, y) = ||x - y|| = |x - y|$
- Geometrical interpretation on  $\mathbb{R}^2$   
 $d(x, 0) = ||x - 0|| = ||x||$   
 $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
- Geometrical interpretation on  $\mathbb{R}^3$   
 $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$



**Properties**

- $d(x, y) \geq 0, d(x, y) = 0 \Leftrightarrow x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y), \forall z \in \mathbb{R}^n$  (Triangle inequality)  
 $d(x, y) = d(x, z) + d(z, y) \Leftrightarrow z$  is a convex linear combination of  $x$  and  $y$ :  $\exists d \in [0, 1]: z = ax + (1 - a)y$   
 $(z$  is a point of the line segment between  $x$  and  $y$ )

**Proof**

- $d(x, y) = ||x - y|| \geq 0 \Rightarrow d(x, y) \geq 0, \forall x, y \in \mathbb{R}^n$   
 $(d(x, y) = 0 \Leftrightarrow x = y) \equiv (d(x, y) = 0 \Rightarrow x = y) \wedge (x = y \Rightarrow d(x, y) = 0)$ 
    - $d(x, y) \Rightarrow x = y$ . Assume that  $d(x, y) = 0$ , that is  $\sqrt{\sum_{i=1}^n (x_i + y_i)^2} = 0 \Rightarrow \sum_{i=1}^n (x_i + y_i)^2 = 0 \Rightarrow \forall i, x_i - y_i = 0 \Rightarrow \forall i, x_i = y_i \Rightarrow x = y$
    - $x = y \Rightarrow d(x, y) = 0$   
 Assume that  $x=y$ . We have  
 $d(x, y) = ||x - y|| = ||x - x|| = ||0|| = 0$
  - $d(x, y) = ||x - y|| = |-(y - x)| = |-1| * ||y - x|| = ||y - x|| = d(y, x)$
  - $d(x, y) = ||x - y|| = ||(x - z) + (z - y)|| \leq ||x - z|| + ||z - y|| = d(x, z) + d(z, y)$
- NB  
 $d(x, y) = d(x, z) + d(z, y) \Rightarrow \exists a \in [0, 1]: z = ax + (1 - a)y$   
 $\exists a \in [0, 1]: z = ax + (1 - a)y \Rightarrow d(x, y) = d(x, z) + d(z, y)$

We introduce notion of neighborhood of a point  $x^\circ \in \mathbb{R}^n$   
**Definition:**

$x^0 \in R^n$  and  $e \in R$  such that  $e > 0$ . The subset of  $R^n$  ( $B_e(x^0)$ ) defined by  $B_e(x^0) = \{x \in R^n: |x - x^0| < e\} = \{x \in R^n: d(x, x^0) < e\}$  is called the **neighborhood** of  $x^0$  with radius  $e > 0$

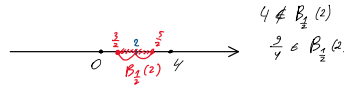
Eg.

$$x^0 = 2, e = \frac{1}{2}$$

$$B_{\frac{1}{2}}(2) = \left\{x \in R: |x - 2| < \frac{1}{2}\right\} = \left(\frac{3}{2}, \frac{5}{2}\right)$$

NB.

$$|x - 2| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x - 2 < \frac{1}{2} \Rightarrow 2 - \frac{1}{2} < x < 2 + \frac{1}{2} \Rightarrow \frac{3}{2} < x < \frac{5}{2}$$



NB.

a.  $n = 1, B_\epsilon(x^0) = \{x \in R: |x - x^0| < \epsilon\}$   
 $|x - x^0| < \epsilon \Rightarrow -\epsilon < x - x^0 < \epsilon \Rightarrow x^0 - \epsilon < x < x^0 + \epsilon \Rightarrow x \in (x^0 - \epsilon, x^0 + \epsilon); B_\epsilon(x^0) = (x^0 - \epsilon, x^0 + \epsilon)$

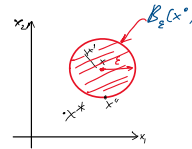
EG.  $B_2(5) = (3, 7)$  - open interval

b.  $n = 2: B_\epsilon(x^0) = \{x \in R^2: |x - x^0| < \epsilon\}$

$$|x - x^0| < \epsilon \Rightarrow \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2} < \epsilon \Rightarrow (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 < \epsilon^2$$

c. On  $R$  we can define other two types of neighborhood:

- I.  $[x_0, x_0 + \epsilon)$  - Right neighborhood of  $x_0$  and radius  $\epsilon > 0$
- II.  $(x_0 - \epsilon, x_0]$  - Left neighborhood of  $x_0$  and radius  $\epsilon > 0$



Set of all the points inside a circle with center  $(x_1^0, x_2^0)$

NB.

$A = [2, 10)$   
 $\forall x \in A, x \leq a$  with  $x \in [10, +\infty)$   
 $\forall x \in A, x \leq 10$  and 10 is the least upper bound of  $A$ .

*proof strategy*  
 $p \Rightarrow q \wedge r \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$

**Proposition:**

Let  $A \subseteq R$  be a set bounded from above. We have  $a = \sup A \Leftrightarrow \begin{cases} \forall x \in A, x \leq a \\ \forall \epsilon > 0, \exists x \in A: x > a - \epsilon \end{cases}$

**Proof**

( $\Rightarrow$ )

Assume that  $a = \sup A$ . Therefore  $a$  is the least upper bound of  $A$  and  $\forall x \in A, x \leq a$   
 Suppose that  $\forall \epsilon > 0, \exists x \in A: x > a - \epsilon$  is false, that is  
 $\exists \epsilon > 0: \forall x \in A, x \leq a - \epsilon$

but  $x \leq a - \epsilon < a$  therefore  $a$  is not the least upper bound of  $A$ , therefore our supposition is false and initial conclusion is true.

( $\Leftarrow$ )

Assume that  $\forall x \in A, x \leq a$  and  $\forall \epsilon > 0, \exists x \in A: x > a - \epsilon$ .  
 We suppose that  $a$  is not  $\sup A$ . Let  $b \in R$  be the least upper bound of  $A$ , therefore  $b < a$   
 $b < a \Rightarrow b < a - \epsilon + \epsilon \Rightarrow b < x + \epsilon \Rightarrow b - \epsilon < x \Rightarrow b < x$   
 Therefore the supposition is false and  $a = \sup A$

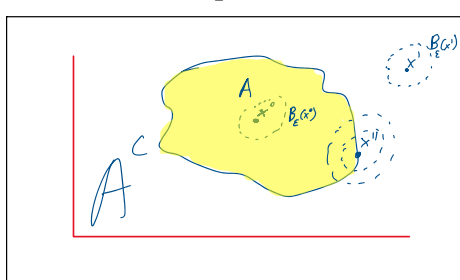
**Proposition**

Let  $A \subseteq R$  be a set bounded from below. Then

$$b = \inf A \Leftrightarrow \begin{cases} \forall x \in A, b \leq x \\ \forall \epsilon > 0, \exists x \in A: x < b + \epsilon \end{cases}$$

**Do at home**

**Classification of points of a set  $A$  w.r.t.  $R^n$ .**



- $x^0$  - Interior point of  $A$
- $x^1$  - Exterior point of  $A$
- $x''$  - Boundary point of  $A$

**Definition**

$A \subseteq R^n, x^0 \in R^n$  is an **interior point** of  $A$  if a with radius  $\epsilon > 0$  exists such that  $B_\epsilon(x^0) \subseteq A$

Eg.

$$A = (2, 10), x^0 = 5, B_1(5) = (4, 6) \subseteq (2, 10) = A$$

NB.:

- $x^0$  is an interior point of  $A \Rightarrow x^0 \in A$
- $x^0 \notin A \Rightarrow x^0$  is not an interior point of  $A$

**Definition**

$A \subseteq R^n, x^0 \in R^n$  is an **exterior point** of  $A$  if a radius  $\epsilon$  exists such that  $B_\epsilon(x^0) \subseteq A^c$

**Definition**

$A \subseteq R^n, x^0 \in R^n$  is a **boundary point** of  $A$  if  $x^0$  is neither an interior nor exterior point of  $A$

Eg.  
 $A = [2,10], x^0 = 5$  interior point;  $x' = 12$  – exterior point



NB.

$A \subseteq \mathbb{R}^n, x^0 \in \mathbb{R}^n$  is a boundary point of  $A$  if  $\forall \epsilon > 0$  we have  $B_\epsilon(x^0) \cap A \neq \emptyset$  and  $B_\epsilon(x^0) \cap A^c \neq \emptyset$   
 that is we have points of  $A$  and  $A^c$

- $\text{int}A$  = Set of all interior points of  $A$  (**Interior of  $A$** )
- $dA$  = Set of all boundary points of  $A$  (**Boundary or frontier of  $A$** )

Eg.

$A = [2,15]$   
 $\text{int} A = (2,15)$   
 $dA = \{2,15\}$   
 $\text{ext} A = (-\infty, 2) \cup (15, +\infty)$

### Lemma 135

Special case:

$A \subseteq \mathbb{R}$  bounded set. Then  $\sup A \in dA$  and  $\inf A \in dA$

Eg.

$A = (2,15]$  bounded  $dA = \{2,15\}$   
 $\sup A = 15 \in dA; \inf A = 2 \in dA$

Def

$A \subseteq \mathbb{R}^n, x^0 \in A$  is called an **isolated point** of  $A$  if  $B_\epsilon(x^0)$  exists such that  $B_\epsilon(x^0) \cap A = \{x^0\}$

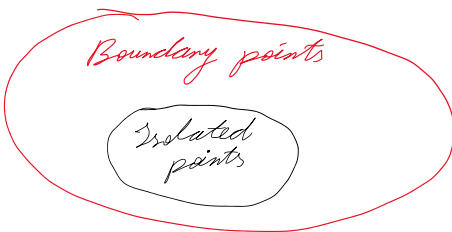
Eg.

$A = \{-2\} \cup [0,1] \cup \{15\} \subseteq \mathbb{R}$ . Bounded subset of  $\mathbb{R}$ :  
 $B_{0,5}(-2) \cap A = \{-2\}, x^0 = -2$  – isolated point of  $A$   
 $B_1(15) \cap A = \{15\}, x^0 = 15$ . Isolated point of  $A$ .  
 $dA = \{-2,0,1,15\}, \inf A = (0,1)$

### Lemma 138

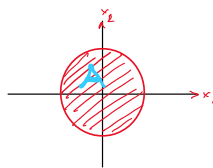
Isolated points are boundary points.

NB. Boundary point  $\neq$  Isolated points



Eg.

$A = \{x \in \mathbb{R}^2: x_1^2 + x_2^2 \leq 9\} \subseteq \mathbb{R}^2$   
 $dA = \{x \in \mathbb{R}^2: x_1^2 + x_2^2 = 9\}$   
 $\text{int}A = \{x \in \mathbb{R}^2: x_1^2 + x_2^2 < 9\}$



Definition

$A \subseteq \mathbb{R}^n, x^0 \in \mathbb{R}^n$  is called a **limit (of an accumulation) point** of  $A$  if any  $B_\epsilon(x^0)$  contains at least one point of  $A$  different from  $x^0$ .

$$\left\{ \begin{array}{l} x^0 \in \mathbb{R}^n \left\{ \begin{array}{l} \text{Interior point of } A \\ \text{Exterior point of } A; x^0 \in \mathbb{R}^n \\ \text{Boundary point of } A \end{array} \right. \left\{ \begin{array}{l} \text{Isolated point } x^0 \in A \\ \text{or} \\ \text{Limit/Accumulation point of } A \end{array} \right. \end{array} \right.$$
 We can classify any point  $x^0 \in \mathbb{R}^n$  w.r.t.  $A$

Eg.

$A = (0,10]$   
 $x^0 = 5 \rightarrow$  interior point and accumulation point  
 $x^0 = 0 \rightarrow$  boundary point and accumulation point  
 $x^0 = 10 \rightarrow$  boundary point and accumulation point  
 $dA = \{0,10\}, \inf A = (0,10), A' = [0,10]$

The set of all the limit points of  $A$  ( $A'$ ) is called the **derived set of  $A$** .

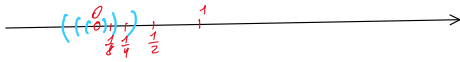
### Lemma 140

$A \subseteq \mathbb{R}^n$

- $x^0$  interior point of  $A \Rightarrow x^0$  is a limit point of  $A$ .  
 $\text{int}A \subseteq A'$
- $x^0 \in dA$  is a limit point  $\Leftrightarrow x^0$  is not an isolated point

Eg.

$$A = \left\{ \frac{1}{n} : n \geq 1 \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$$



0 is an accumulation point of  $A$  ( $0 \notin A$ )  
In  $B_\epsilon(0)$  we find  $\infty$  points of  $A$ .

### Proposition 143

Each  $B_\epsilon(x^0)$  of the limit point  $x^0$  of  $A$  contains infinitely many points of  $A$

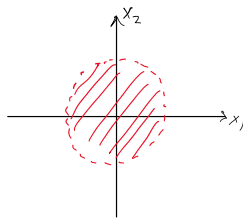
Def.

$A \subseteq \mathbb{R}^n$ . Set  $A$  is called an **open set** of  $\mathbb{R}^n$  if all its points are interior ones, i.e.  $\text{int}A = A$

Eg.

$$A = (-2, 10) \text{ open subset of } \mathbb{R}: (\text{int}(-2, 10) = (-2, 10) = A)$$

$$A = \{x \in \mathbb{R}^2: |x| < 1\} \equiv \{x \in \mathbb{R}^2: x_1^2 + x_2^2 < 1\}$$



$$\text{int}A = A \Rightarrow A \text{ open subset of } \mathbb{R}^2$$

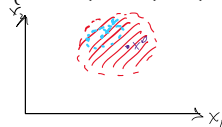
$$dA = \{x \in \mathbb{R}^2: |x| = 1\}$$

NB.

$$B_\epsilon(x^0) = \{x \in \mathbb{R}^n: |x - x^0| < \epsilon\}, \epsilon > 0, x^0 \in \mathbb{R}^n$$

In case  $n=2$

$$B_\epsilon(x^0) = \{x \in \mathbb{R}^2: (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 < \epsilon^2\}$$



$B_\epsilon(x^0)$  is an open subset of  $\mathbb{R}^n$  ( $B_\epsilon(x^0)$  is made up of interior points)

### Lemma 147

Neighborhoods are open sets

*Proof*

Let  $x$  be any point of  $B_\epsilon(x^0)$ , that is  $d(x^0, x) < \epsilon$ .

Then  $0 < \epsilon - d(x^0, x)$  we have  $0 < \epsilon_x \leq \epsilon - d(x, x^0)$   
*Small n-hood of x*

Where  $\epsilon_x$  is a radius of a neighborhood with center  $x$

We have:

$$\forall y \in B_{\epsilon_x}(x), d(x, y) < \epsilon_x \leq \epsilon - d(x, x^0)$$

Therefore

$$d(y, x^0) \leq d(y, x) + d(x, x^0) < d(x, x^0) + \epsilon - d(x, x^0) = \epsilon$$

Which implies that  $d(y, x^0) < \epsilon$  and  $y \in B_\epsilon(x^0)$ . We have

$$\forall y \in B_{\epsilon_x}(x), y \in B_{\epsilon_x}(x) \Rightarrow y \in B_\epsilon(x^0) \text{ and } B_{\epsilon_x}(x) \subseteq B_\epsilon(x^0) \text{ and } B_\epsilon(x^0) \text{ is open}$$

The set  $A \cup dA$  is called the **closure** of  $A$  ( $\bar{A}$ ), that is  $\bar{A} = A \cup dA$

NB.

$$A \subseteq \bar{A} \text{ because } \bar{A} = A \cup dA$$

Eg.

$$A = \{-2\} \cup (2, 10) \quad \text{--- } dA \quad \quad \quad A \subseteq \bar{A}$$

$$\bar{A} = [\{-2\} \cup (2, 10)] \cup \{-2, 2, 10\} = \{-2\} \cup [2, 10]$$

Def.

$A \subseteq \mathbb{R}^n$ .  $A$  is said to be a **closed** set of  $A$  contains all its boundary points, i.e.  $\bar{A} = A$

Eg.

$$A = (0, 10) \rightarrow \text{open}$$

$$B = [0, 10] \rightarrow \text{closed}, \bar{B} = B$$

### Theorem

$A \subseteq \mathbb{R}^n$ .  $A$  is open if and only if  $A^c$  is closed

(only one direction)  
 **$A$  open  $\Rightarrow A^c$  closed**

Proof: Assume that  $A$  is open, that is  $\text{int}A = A$ .  $A^c$  is closed when  $dA^c \subseteq A^c$  ( $x \in dA^c \Rightarrow x \in A^c$ ).

Let  $x \in dA^c$ , we have

$$x \in dA^c \Rightarrow x \notin \text{int}A \Rightarrow x \notin A \Rightarrow x \in A^c$$

Therefore  $dA^c \subseteq A^c$ . and  $A^c$  is closed

### Proposition 158

$$\bar{A} = A \cup A'$$

Eg.

$$A = (2,15), A' = [2,15], A \cup A' = (2,15) \cup [2,15] = [2,15]$$

Eg.

$$A = \left\{ \frac{1}{n^2} : n \geq 1 \right\} = \left\{ 1, \frac{1}{4}, \frac{1}{9}, \dots \right\}$$

Is  $A$  open? Is  $A$  closed?

- $\text{int}A = \emptyset \neq A$ ,  $A$  is not an open subset of  $\mathbb{R}$
- $dA = \{0\}$ ,  $0 \notin A$ ,  $A$  is not a closed subset of  $\mathbb{R}$

### Corollary 159

$A \subseteq \mathbb{R}^n$ .  $A$  is closed  $\Leftrightarrow A$  contains all its accumulation points

Eg.

$$A = \left\{ \frac{1}{n^2} : n \geq 1 \right\} = \left\{ 1, \frac{1}{4}, \frac{1}{9}, \dots \right\}$$

$0 \notin A$ ,  $0$  is an accumulation point of  $A \Rightarrow A$  is not closed

**Proof**

Let  $A$  be a subset of  $\mathbb{R}^n$ . Assume that  $A$  is closed, that is  $A = \bar{A}$ . We have to show that  $A' \subseteq A$ .

( $x \in A' \Rightarrow x \in A$ ), so

$$x \in A' \Rightarrow x = A' \cup A \Rightarrow x \in \bar{A} \Rightarrow x \in A \quad \text{and} \quad A' \subseteq A$$

Assume that  $A' \subseteq A$ , we have to show that ( $\bar{A} = A$ , that is  $\bar{A} \subseteq A$  and  $A \subseteq \bar{A}$ )

$$A \subseteq \bar{A}: x \in A \Rightarrow x \in A \cup A' \Rightarrow x \in \bar{A}$$

$$\bar{A} \subseteq A: x \in \bar{A} \Rightarrow x \in A \cup A' \Rightarrow x \in A$$

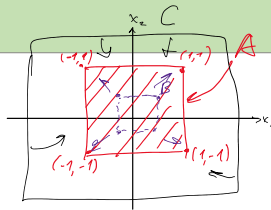
Therefore,  $A = \bar{A}$  and  $A$  is closed

$\text{int}A \subseteq A \subseteq \bar{A}$

Eg.

$$A = \{x \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}$$

$A$  is closed



### Proposition 161

Given  $A \subseteq \mathbb{R}^n$ . We have

- $\text{int}A$  is the largest **open set** contained in  $A$
- $\bar{A}$  is the smallest **closed set** which contains  $A$

### Corollary 163

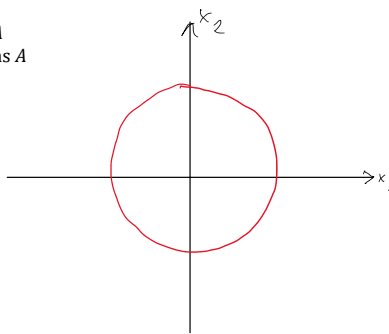
The boundary of a set of  $\mathbb{R}^n$  is a closed set

Eg.

$$B = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$$

$\text{int}B = \emptyset$  ( $B$  is not open)

$dB = B$ ,  $B$  is closed



### Proposition 164

For any set  $A \subseteq \mathbb{R}^n$ ,  $dA = \bar{A} \rightarrow \text{int}A$

NB.  $A$  is open  $\Leftrightarrow dA \cap A = \emptyset$

## Set stability

Eg.

$$A = [2,10] \cup [12,20],$$

$$B = (-1,0) \cup (2,10)$$

$A$  is closed because is the union of two closed sets

$B$  is open because is the union of two open sets

### Theorem 165

Intersection of **finite** collection of open sets is also open

Eg.

$$\left\{ \begin{array}{l} B = (-1,0) \cup (2,10) \\ B \text{ is a finite intersection of open sets} \end{array} \right.$$

The union of any collection of open sets is open.

NB:  $A_1, A_2, \dots, A_n, \dots \rightarrow$  open sets

$\bigcap_{i=1}^n A_i$  is open and  $\bigcup_{i=1}^{+\infty} A_i$  is open - (Taking  $A_{n+1} = A_{n+2} = \dots = \emptyset$ ,  $\bigcup_{i=1}^n A_i$  is open)

### Corollary 166

The union of a finite collection of closed sets is closed, the intersection of any collection (finite or not) of closed sets is also closed.

NB.

$A_1, A_2, \dots, A_n \dots$  closed sets  
 $\left\{ \bigcup_{i=1}^n A_i \text{ is closed} \right.$   
 $\left\{ \bigcap_{i=1}^{\infty} A_i \text{ is closed} \right.$

### Compact set

Definition

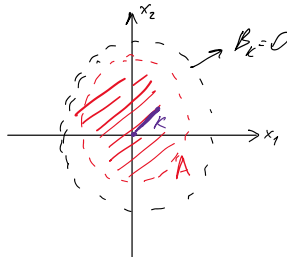
$A \subseteq \mathbb{R}^n$ .  $A$  is said to be **bounded** if a real number  $k > 0$  exists such that  $\|x\| < k, \forall x \in A$ .

NB.

$B_k(0) = \{x \in \mathbb{R}^n : \|x\| < k\}$ , Height of 0

Eg.

$A = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 9\}$



### Proposition 169

A set  $A$  is bounded  $\Leftrightarrow k$  exists such that  
 $\forall x \in A, |x_i| < k, \forall i = 1, \dots, n$

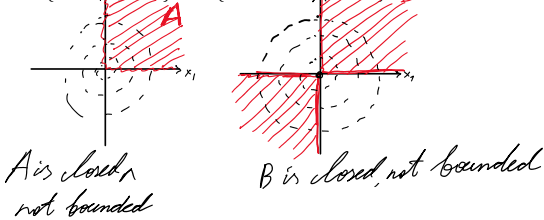
Eg.

$A = [-2, 0] \cup (1, 10)$  – bounded

$B = [-2, 0] \cup [1, 10]$  – bounded

Eg.

$A = \{x \in \mathbb{R}^2 : x_1 \geq 0\}; B = \{x \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$



Def.

$A \subseteq \mathbb{R}^n$ .  $A$  is said to be a compact subset of  $\mathbb{R}^n$  if  $A$  is **closed** and **bounded**  
 $A$  compact  $\Leftrightarrow A$  closed  $\wedge$   $A$  bounded

Eg.

$B = [0, 2] \cup [5, 12]$

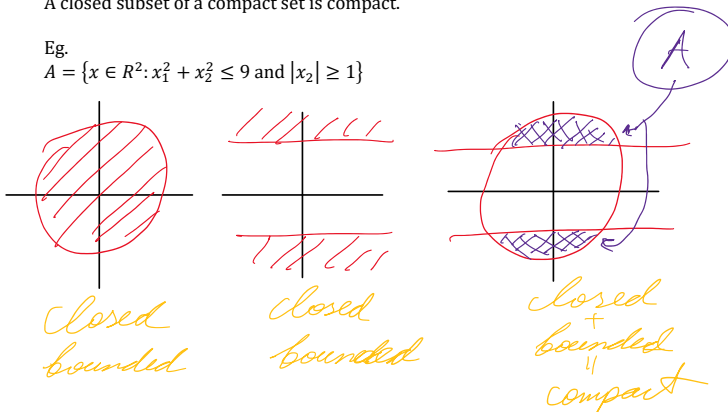
$B$  is closed and bounded, hence  $B$  is compact subset of  $\mathbb{R}$

### Proposition 173

A closed subset of a compact set is compact.

Eg.

$A = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 9 \text{ and } |x_2| \geq 1\}$



# Chapter 6 - Real functions

giovedì 21 settembre 2023 15:56

Def.

$A, B$  non-empty sets. A function  $f$  defined on  $A$  and with values in  $B$  ( $f: A \rightarrow B$ ) is a rule which associates to each element of  $A$  one and only one element of  $B$ .

$A$  - domain of  $f$

$B$  - codomain of  $f$

NB.

$$A \subseteq \mathbb{R}^n$$

$$B \subseteq \mathbb{R}^m, \text{ (but for us, } B = \mathbb{R}^m)$$

$$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m \mid n, m - \text{positive integers}$$

$$1. n = m = 1: f: A \subseteq \mathbb{R} \rightarrow \mathbb{R} \text{ (One-variable real functions)}$$

$$2. m = 1: f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \text{ (Multivariable functions)}$$

$$3. n = 1: f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}^m \text{ (Curves of } \mathbb{R}^n)$$

$$4. m, n \geq 2: f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ (Operators)}$$

NB.

$f(x)$  the image of  $x$  under  $f$

$f(A)$  the image set (or the image of  $A$  under  $f$ )

Eg.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 + 2, A = \mathbb{R}, f(\mathbb{R}) = [2, +\infty)$$

$$f(\mathbb{R}) \subset \mathbb{R}, n = -1 \in \mathbb{R}, f(-1) = (-1)^2 + 2 = 3$$

Eg.

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}, f(x) = \sqrt{x_1} + \ln x_2 + \frac{1}{x_3}$$

$$A = \{x \in \mathbb{R}^3: x_1 \geq 0, x_2 > 0, x_3 \neq 0\}, \exists f(x)$$

$$x^* = \left(4, e^2, \frac{1}{2}\right) \in A$$

$$\exists f(x^*); f\left(4, e^2, \frac{1}{2}\right) = \sqrt{4} + \ln e^2 + \frac{1}{0.5}$$

Eg.

$$f: \mathbb{R} \rightarrow \mathbb{R}^2, f(x) = (x, 2x^2)$$

$$x = 1, f(1) = (1, 2);$$

$$x = -1, f(-1) = (-1, 2)$$

Eg.

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$f(x_1, x_2) = (x_1 + x_2, 2x_2, x_1 - x_2^2)$$

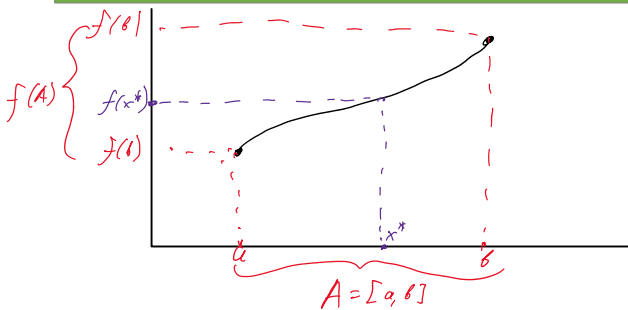
$$x^* = (-1, 2)$$

$$f(x^*) = f(-1, 2) = (1, 4, -5) \in \mathbb{R}^3$$

**Operator** from  $\mathbb{R}^2$  to  $\mathbb{R}^3$

## Definition - one-variable real functions

The graph of  $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$  ( $G_f$ ) is the set  $G_f = \{(x, f(x)) \in \mathbb{R}^2: x \in A\} \subseteq A \times \mathbb{R}$ .

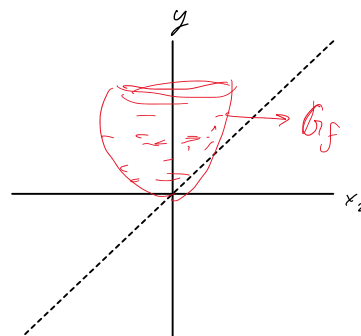


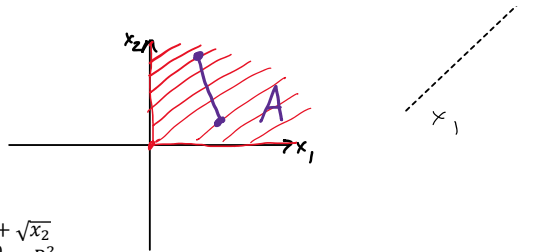
NB.

$f: A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ . The graph of  $f$  ( $G_f$ ) is a subset of  $\mathbb{R}^3$  defined by

$$\begin{cases} G_f = \{(x_1, x_2, y) \in \mathbb{R}^3: y = f(x_1, x_2) \text{ and } (x_1, x_2) \in A\} \\ G_f \subseteq \mathbb{R}^3 \end{cases}$$

Is called the subspace of  $f$





Eg.

$$f: A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}, f(x) = \sqrt{x_1} + \sqrt{x_2}$$

$$A = \{x \in \mathbb{R}^2: x_1 \geq 0, x_2 \geq 0\} = \mathbb{R}_+^2$$

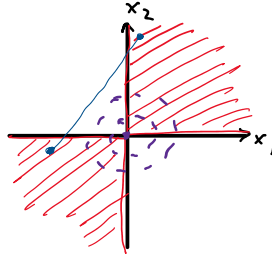
- A is closed and not bounded
- A convex ( $\forall x^1, x^2 \in A, \forall a \in [0,1], (1-a)x^1 + ax^2 \in A$ )

Eg.

$$f: A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}, f(x) = \sqrt{x_1 x_2}$$

$$A = \{x \in \mathbb{R}^2: x_1 x_2 \geq 0\}$$

- A is closed and not bounded
- A is not convex



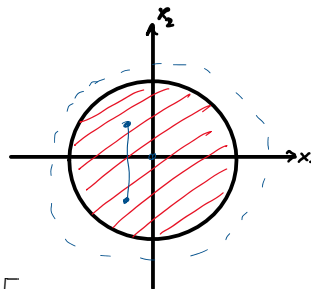
Eg.

$$g: A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R},$$

$$g(x) = \sqrt{1 - x_1^2 - x_2^2}$$

$$A = \{x \in \mathbb{R}^2: 1 - x_1^2 - x_2^2 \geq 0\} = \{x \in \mathbb{R}^2: x_1^2 + x_2^2 \leq 1\}$$

- A is closed and bounded  $\Leftrightarrow$  A compact
- A is convex



$$x^* = \left(\frac{1}{2}, \frac{1}{2}\right) \in A, \exists g\left(\frac{1}{2}, \frac{1}{2}\right) = \sqrt{1 - \frac{1}{4} - \frac{1}{4}} = \sqrt{\frac{1}{2}}$$

## Preimages and level curves

NB.

$$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, y \in \mathbb{R}, f^{-1}(\{y\}) = \{x \in A: y = f(x)\}$$

$f^{-1}(\{y\})$  is called the preimage of  $y$  under  $f$

Consider  $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  ( $x \in A, x = (x_1, x_2, \dots, x_n), y \in \mathbb{R}$ )

The preimage of  $y \in \mathbb{R}$  under  $f$  is a subset of  $A$  given by  $f^{-1}(\{y\}) = \{x \in A \subseteq \mathbb{R}^n: y = f(x)\}$

Eg.

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x_1, x_2) = x_1^2 + x_2^2,$$

$$f^{-1}(\{0\}) = \{(0,0)\}$$

$0 \in \mathbb{R}^2$  is the preimage of  $0 \in \mathbb{R}$  under  $f$

The preimage of a subset  $\Delta \subseteq \mathbb{R}$  under  $f$  is a subset of  $A$  given by

$$f^{-1}(\Delta) = \{x \in A: y = f(x) \text{ and } y \in \Delta\}$$

NB.

Let  $K$  be a real number.  $f^{-1}(\{k\})$  of  $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be the **level curve** or **level set** of  $f$  with level  $K$ .

$$f^{-1}(\{k\}) \subseteq \mathbb{R} = \text{Set of points of } A \text{ where function } f \text{ takes on the same value } K$$

Eg.

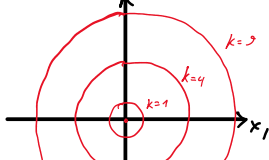
$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x) = x_1^2 + x_2^2$$

Find the level curve of  $f$  with level  $k \in \mathbb{R}$

$$f^{-1}(\{k\}) = \{x \in \mathbb{R}^2: x_1^2 + x_2^2 = k \text{ and } k \in \mathbb{R}\}$$

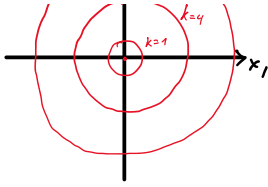
- $k < 0: f^{-1}(\{k\}) = \emptyset$
- $k = 0: f^{-1}(\{k\}) = \{(0,0)\}$

$x_1^2 + x_2^2 = (\sqrt{k})^2$   $f^{-1}(\{k\})$  set of all the circumference with center  $(0,0)$  and radius  $\sqrt{k}$



$$k=9: x_1^2 + x_2^2 = 2^2$$

All the points on the circumference with center  $(0,0)$  and radius 2 have the same value 4



All the points on the circumference with center (0,0) and radius 2 have the same value 4

## Algebra of functions

$$f: A_f \subseteq R \rightarrow R, g: A_g \subseteq R \rightarrow R, A_f \cap A_g = A \neq \emptyset$$

NB.

$$\begin{cases} R^A = \text{Set of all the functions } f: A \subseteq R^n \rightarrow R \\ f, g \in R^A \end{cases}$$

Taking  $f, g \in R^A$  we can build new functions

1.  $f + g \in R^A, \forall x \in A, (f + g)(x) = f(x) + g(x)$
2.  $af \in R^A, \forall x \in A, (af)(x) = a * f(x), a \in R$
3.  $f * g \in R^A, \forall x \in A, (f * g)(x) = f(x) * g(x)$
4.  $\frac{f}{g} \in R^A, \forall x \in A, \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$  with  $g(x) \neq 0$

NB.

$$a * f + b * g \in R^A, \forall x \in A (af + bg)(x) = a * f(x) + b * g(x)$$

with  $a, b \in R$

Eg.

$$f, g \in R^A, f(x) = x_1^2 + x_2^2 + 1, g(x) = x_1^2 - x_2^2$$

$$1. f + 2g \in R^A (R^{R^2}), (f + 2g)(x) = x_1^2 + x_2^2 + 1 + 2(x_1^2 - x_2^2) = 3x_1^2 - x_2^2 + 1$$

$$(f + 2g)(-1, 1) = 3(-1)^2 - 1^2 + 1 = 4 - 1 = 3$$

$$2. \frac{f}{g} \in R^{R^2}: \left(\frac{f}{g}\right)(x) = \frac{x_1^2 + x_2^2 + 1}{x_1^2 - x_2^2} \text{ with } x_1^2 \neq x_2^2$$

$$\frac{f}{g}(2, 1) = \frac{4 + 1 + 1}{4 - 1} = \frac{6}{3} = 2$$

## Composition of functions

Given

$$f: A_f \subseteq R^n \rightarrow R, g: A_g \subseteq R^n \rightarrow R, f(A_f) \subseteq A_g$$

We can define a new function

$$g \circ f: A_f \subseteq R^n \rightarrow R, g \circ f(x) = g[f(x)]$$

Is said to be "g composite f"

1.  $g \circ f \neq f \circ g$
2.  $f \circ (g \circ h) = (f \circ g) \circ h$
3.  $f \circ f = f^2$

Eg.

$$f: R^2 \rightarrow R, f(x) = x_1^2 + x_2^2$$

$$g: R \rightarrow R, g(t) = 2t + 1$$

$$f(R^2) = [0, +\infty) \subseteq R = A_g, g \circ f: R^2 \rightarrow R$$

$$g \circ f(x) = g[f(x)] = g[x_1^2 + x_2^2] = 2(x_1^2 + x_2^2) + 1$$

$$g \circ f(-1, 2) = 2((-1)^2 + 2^2) + 1 = 7$$

## Injective, Surjective, and Bijective functions

### Injective functions

$f: A \subseteq R^n \rightarrow R$ ,  $f$  is said to be injective if  $\forall y \in R$  at most one preimage  $x \in A$  exists (under  $f$ ) such that  $y = f(x)$

NB.

$f$  is injective when  $\forall x_1, x_2 \in A$

1.  $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
2.  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

*= Only one y for each x*

And  $f$  is not injective if  $\exists x_1, x_2 \in A: x_1 \neq x_2$  and  $f(x_1) = f(x_2)$

Eg.  
 $g: \mathbb{R}^2 \rightarrow \mathbb{R}, g(x) = x_1 + x_2 + 1$ .

Is  $g$  injective?

- $x^* = (-1, 1), g(x^*) = -1 + 1 + 1 = 1$
- $x' = (-2, 2), g(x') = -2 + 2 + 1 = 1$

Hence,  $g$  is not injective

### Surjective function

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ .  $f$  is surjective if  $\forall y \in \mathbb{R}$  at least one preimage  $x \in A$  exists (under  $f$ ) such that  $y = f(x)$

= *y can be any value*

NB.

$f(A) = \mathbb{R} \Leftrightarrow f$  is surjective

Eg.

$g: \mathbb{R}^3 \rightarrow \mathbb{R}, g(x) = \|x\|$

Is  $g$  surjective?

No, because norm is always positive

$\forall y \in \mathbb{R}$  with  $y < 0, g(x) = y$  or  $\|x\| = y$  -- impossible

### Bijjective function

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ .  $f$  is said to be bijective if  $\forall y \in \mathbb{R}$  one and only one preimage  $x \in A$  (under  $f$ ) exists such that  $y = f(x)$

NB.

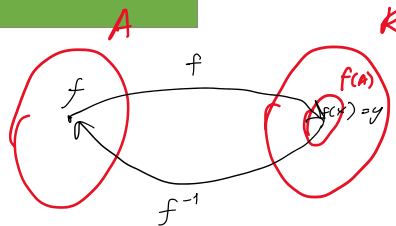
$f$  is bijective if  $f$  is both injective and surjective

### Inverse functions

Let  $f$  be a real function with domain  $A$ . Given  $f(A) \subseteq \mathbb{R}$  if at each element of  $f(A)$  we can associate a unique value  $x \in A$  then we say that  $f$  is invertible function, that is we can define the inverse function  $f^{-1}: f(A) \rightarrow A$  with  $x = f^{-1}(y)$

Properties:

1.  $\forall x \in A, f^{-1} \circ f(x) = x$
2.  $\forall y \in f(A), f \circ f^{-1}(y) = y$



NB.

$f$  injective  $\Leftrightarrow f$  invertible

Eg.

$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^3 + 1$

$f$  is injective  $\Rightarrow f$  is invertible

$$f^{-1}(\{y\}) = \{x \in \mathbb{R}: x^3 + 1 = y \text{ and } y \in \mathbb{R}\}$$

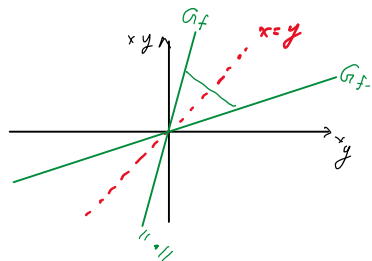
$$= \{x \in \mathbb{R}: x^3 = y - 1 \text{ and } y \in \mathbb{R} = \{\sqrt[3]{y-1}\}\}$$

$$f^{-1}: \mathbb{R} \rightarrow \mathbb{R}, x = f^{-1}(y), \forall y \in \mathbb{R}, f \circ f^{-1}(y) = y$$

$$f[f^{-1}(y)] \Rightarrow (f^{-1}(y))^3 + 1 = y$$

$$\Rightarrow (f^{-1}(y))^3 = y - 1$$

$$\Rightarrow f^{-1}(y) = \sqrt[3]{y-1}$$



$G_f$  and  $G_{f^{-1}}$  are symmetrical w.r.t. the line of equation  $y = x$

### Bounded functions

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ .  $f$  is said to be:

1. Bounded from above if  $\forall x \in A, \exists M \in \mathbb{R}: f(x) \leq M$
2. Bounded from below if  $\forall x \in A, \exists m \in \mathbb{R}: m \leq f(x)$
3. Bounded (from above and below) if  $\forall x \in A, \exists M, m \in \mathbb{R}: m \leq f(x) \leq M$

NB.

$f$  is not bounded  $\Rightarrow f$  is unbounded

Eg.

$f: \mathbb{R}^3 \rightarrow \mathbb{R}, f(x) = 1 - |2x| \leq 1, M \in [1, +\infty) \Rightarrow f$  is bounded from above

$g: \mathbb{R}^3 \rightarrow \mathbb{R}, g(x) = |x_1 - 1| + |x_2 x_3| \geq 0, m \in (-\infty, 0)$

$h: \mathbb{R}^2 \rightarrow \mathbb{R}, h(x) = \frac{1}{x_1^2 + x_2^2 + 1}$

$$h(x) \geq 0, \forall x \in \mathbb{R}^2$$

NB

$$x_1^2 + x_2^2 \geq 0 \Rightarrow x_1^2 + x_2^2 + 1 \geq 1 \Rightarrow \frac{1}{x_1^2 + x_2^2 + 1} \leq 1$$

$$0 \leq \frac{1}{x_1^2 + x_2^2 + 1} \leq 1$$

$m \in (-\infty, 0]$  and  $M \in [1, +\infty)$ , hence  $h$  is bounded

## Lemma 215

Let  $f$  be a real function with  $A \subseteq \mathbb{R}$ .  $f$  is bounded  $\Leftrightarrow$  a real number  $k > 0$  exists such that  $\forall x \in A, |f(x)| \leq k$

Proof

( $\Rightarrow$ )

Assume that  $f$  is bounded, therefore real numbers  $m, M$  exist such that

$$\forall x \in A, m \leq f(x) \leq M$$

Taking a real number  $k > 0$  such that  $-k \leq m \leq f(x) \leq M \leq k$  and we have

$$-k \leq f(x) \leq k \text{ and } \forall x \in A, |f(x)| \leq k$$

( $\Leftarrow$ )

Assume that  $\forall x \in A, |f(x)| \leq k$  with  $k > 0$ . Therefore, we have  $\forall x \in A, -k \leq f(x) \leq k$  and  $f$  is bounded ( $m = -k$  and  $M = k$ )

Eg.

$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x^2$ . Is  $f$  bounded? - No

$f$  is bounded from below ( $m \leq 0$ ) but not from above

Therefore,  $f$  is unbounded

NB.

- $f$  is bounded from below  $\Rightarrow \inf f(x) = \inf f(A)$
- $f$  is bounded from above  $\Rightarrow \sup f(x) = \sup(A)$
- $f$  is bounded from above and below  $\Rightarrow \inf f(x) = \inf f(A) \wedge \sup f(x) = \sup f(A)$   
And  $m \leq \inf f(x) \leq f(x) \leq \sup f(x) \leq M$

Eg.

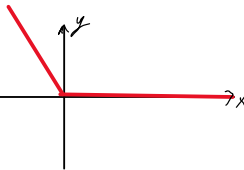
$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \max(x, -x) - x$

$A = \mathbb{R} = (-\infty, +\infty)$

•  $x \geq 0: f(x) = \max(x, -x) - x = x - x = 0$

•  $x < 0: f(x) = \max(x, -x) - x = -x - x = -2x$

$$f(x) = \max(x, -x) - x = \begin{cases} -2x, & x < 0 \\ 0, & x \geq 0 \end{cases}$$



$\sup_{x \in \mathbb{R}} f(x) \nexists (\nexists M) \rightarrow f$  is not bounded

$$\inf_{x \in \mathbb{R}} f(x) = 0 \quad (m \leq 0)$$

$f^{-1}(\{y\})$  - set of all the preimages of  $f \forall y \in \mathbb{R}$ .

- $y < 0: f^{-1}(\{y\}) = \emptyset$
- $y = 0: f^{-1}(\{y\}) = [0, +\infty)$
- $y > 0: f^{-1}(\{y\}) = \{-\frac{y}{2}\}, -2x = y \Rightarrow x = -\frac{y}{2}$

## Monotone functions

Definition

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ .  $f$  is said to be

- **increasing** on  $A$  if  $\forall x_1, x_2 \in A, x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$
- **strictly increasing** on  $A$  if  $\forall x_1, x_2 \in A, x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- **decreasing** on  $A$  if  $\forall x_1, x_2 \in A, x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$
- **strictly decreasing** on  $A$  if  $\forall x_1, x_2 \in A, x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Eg.

$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x + 1$ . Is  $f$  increasing?

$\forall x_1, x_2 \in \mathbb{R}$ , suppose that  $f$  is decreasing.  $x_1 < x_2$  and  $2x_1 + 1 \geq 2x_2 + 1 \Rightarrow$

$$\Rightarrow 2x_1 \geq 2x_2 \Rightarrow x_1 \geq x_2 \text{ - Contradicts initial supposition, hence false}$$

Another strategy?

Assume that  $x_1 < x_2$  with  $x_1, x_2 \in A$ . We have  $f(x_1) = 2(x_1 - x_2 + x_2) + 1$

$$= 2(x_1 - x_2) + 2x_2 + 1 \leq 2x_2 + 1 = f(x_2) - f \text{ is increasing}$$

NB.

- $f$  increasing/decreasing is said to be **monotonic function**
- $f$  strictly increasing/strictly decreasing is said to be **strictly monotonic function**
- $f$  strictly monotonic  $\Rightarrow f$  is monotonic
- Constant functions are simultaneously increasing and decreasing.

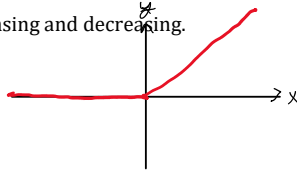
2.  $f$  strictly increasing/strictly decreasing is said to be strictly monotonic function
3.  $f$  strictly monotonic  $\Rightarrow f$  is monotonic
4. Constant functions are simultaneously increasing and decreasing.

Eg.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x + 1$$

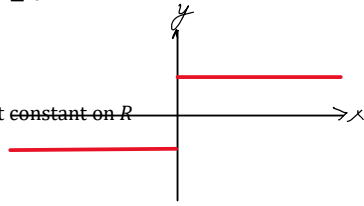
$f$  is strictly increasing  $\Rightarrow f$  is increasing

$$g: \mathbb{R} \rightarrow \mathbb{R}, g(x) = \max(x, 0) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$



Eg.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases} \text{ - Not constant on } \mathbb{R}$$



- $f$  is constant on  $(-\infty, 0)$
- $f$  is constant on  $[0, +\infty)$
- $f$  is not constant on  $\mathbb{R}$

$$f^{-1}(\{1\}) = [0, +\infty);$$

$$f(\mathbb{R}) = \{-1, 1\}$$

$f$  is bounded

**$f$  is increasing**

### Proposition 218

An increasing function  $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is strictly increasing  $\Leftrightarrow f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ ,  $\forall x_1, x_2 \in A$  (if and only if  $f$  is injective)

Proof:

( $\Rightarrow$ )

Assume that  $f$  is strictly increasing and  $f(x_1) = f(x_2)$ . Suppose, by contradiction, that  $x_1 \neq x_2$  ( $x_1 < x_2 \vee x_1 > x_2$ ). We have:

$$x_1 \neq x_2 \Rightarrow x_1 < x_2 \vee x_1 > x_2 \Rightarrow f(x_1) < f(x_2) \vee f(x_1) > f(x_2) \Rightarrow f(x_1) \neq f(x_2)$$

Which contradicts the starting assumption, hence the opposite is true, i.e.,  $x_1 = x_2$  and  $f$  is injective

( $\Leftarrow$ )

Assume that  $\forall x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$  and  $f$  is increasing. We have to show that  $f$  is strictly increasing.

Consider  $x_1, x_2 \in A$  with  $x_1 < x_2$  and

$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \Rightarrow f(x_1) < f(x_2) \vee f(x_1) = f(x_2) \Rightarrow$$

$$\Rightarrow f(x_1) < f(x_2) \text{ or } x_1 = x_2 \Rightarrow f(x_1) < f(x_2)$$

$f$  is strictly increasing on  $A$ .

NB.

1.  $f$  strictly monotonic  $\Rightarrow f$  injective  $\Leftrightarrow f$  invertible
- $$\begin{cases} f \text{ strictly monotonic} \Rightarrow f \text{ invertible} \\ f \text{ invertible} \Rightarrow f \text{ strictly monotonic} \end{cases}$$

Eg.

$$f: \mathbb{R} - \{0\} \rightarrow \mathbb{R}, f(x) = \frac{1}{x} = x^{-1}$$

1.  $f(x) = x^{-1}$  on  $(0, +\infty)$
2.  $f$  is odd
3.  $f$  is strictly decreasing on  $(-\infty, 0)$  and  $(0, +\infty)$
4.  $f$  is not monotonic on  $\mathbb{R}$
5.  $f$  is injective  $\Rightarrow f$  is invertible

NB.

We have that  $f(x_1) = f(x_2)$  if  $x_1 = x_2$ . Therefore we can say:

$$\begin{cases} \forall x_1, x_2 \in A, x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2) \\ \quad \quad \quad f \text{ is increasing} \\ \forall x_1, x_2 \in A, x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2) \\ \quad \quad \quad f \text{ is decreasing} \end{cases}$$

The converse implication ( $p \Rightarrow q; q \Rightarrow p$ ) from the definition of increasing function is

$$\forall x_1, x_2 \in A, f(x_2) \geq f(x_1) \Rightarrow x_1 \geq x_2 (*)$$

The contrapositive of this implication is:

$$\forall x_1, x_2 \in A, x_2 < x_1 \Rightarrow f(x_2) < f(x_1) (**)$$

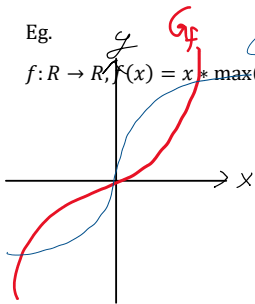
and this is the definition of strictly increasing function

Implications (\*) and (\*\*) are equivalent

### Proposition 220

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ .  
 $f$  is strictly increasing  $\Leftrightarrow \forall x_1, x_2 \in A, f(x_2) \geq f(x_1) \Rightarrow x_2 \geq x_1$

**Proposition 221**  
 $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$   
 $f$  is strictly increasing  $\Leftrightarrow \forall x_1, x_2 \in A, x_1 \leq x_2 \Leftrightarrow f(x_1) \leq f(x_2)$



Eg.  
 $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x * \max(x, -x) = x * |x| = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$

- $f$  is strictly increasing on  $\mathbb{R} \Rightarrow f$  is invertible ( $f^{-1}: \mathbb{R} \rightarrow \mathbb{R}$ )
- $f$  is unbounded and odd
- $f$  strictly increasing  $\Leftrightarrow f^{-1}$  strictly increasing

**Proposition 222**  
 $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ .  
 $f$  is strictly increasing (decreasing)  $\Leftrightarrow$  its inverse  $f^{-1}$  is strictly increasing (decreasing)

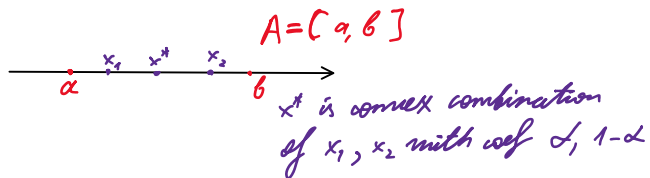
Proof

( $\Rightarrow$ )  
 Assume that  $f$  is strictly increasing ( $f$  is invertible) and  $f^{-1}: f(A) \rightarrow A$ .  
 Let  $z_1, z_2 \in f(A)$  with  $z_2 > z_1$ . We have that  $x_1, x_2 \in A$  exists such that  
 $x_1 = f^{-1}(z_1)$  and  $x_2 = f^{-1}(z_2)$ . Suppose, by contradiction, that  
 $f^{-1}(z_1) \geq f^{-1}(z_2)$ .  
 Since  $f$  is strictly increasing, we have:  
 $f^{-1}(z_1) \geq f^{-1}(z_2) \Rightarrow f[f^{-1}(z_1)] \geq f[f^{-1}(z_2)] \Rightarrow z_1 \geq z_2 \Rightarrow f(x_1) \geq f(x_2)$   
 and we have a contradiction, therefore  $f^{-1}$  is strictly increasing

( $\Leftarrow$ )  
 Assume that  $f^{-1}$  is strictly increasing and  $f: A \rightarrow \mathbb{R}$ . Let  $x_1, x_2 \in A$  with  $x_2 > x_1$   
 We know that  $z_1, z_2 \in f(A)$  exist such that  $z_1 = f(x_1)$  and  $z_2 = f(x_2)$   
 Suppose, by contradiction, that  $f(x_1) \geq f(x_2)$ . Since  $f^{-1}$  is strictly increasing  
 $f(x_1) \geq f(x_2) \Rightarrow f^{-1}[f(x_1)] \geq f^{-1}[f(x_2)] \Rightarrow f^{-1}(z_1) \geq f^{-1}(z_2)$   
 and we have a contradiction with the starting premise, so  $f$  is strictly increasing

NB.

1. Any interval of  $\mathbb{R}$  is a convex set.
2.  $\forall x_1, x_2 \in A, \forall a \in [0,1], x^* = ax_1 + (1-a)x_2$



$\forall x_1, x_2 \in \mathbb{R}, \forall a \in \mathbb{R}, x^* = ax_1 + (1-a)x_2$



## Convex and concave functions

NB:

1.  $x^1, x^2 \in A, a \in [0,1], x^* = (1-a)x^1 + ax^2$   
 $x^*$  is said to be a convex linear combination of  $x^1, x^2$  with coef.  $1-a, a$
2.  $A \subseteq \mathbb{R}^n$  is said to be convex when  
 $\forall x^1, x^2 \in A, \forall a \in [0,1], (1-a)x^1 + ax^2 \in A$
3.  $A \subseteq \mathbb{R}, A$  convex  $\Leftrightarrow A$  is interval

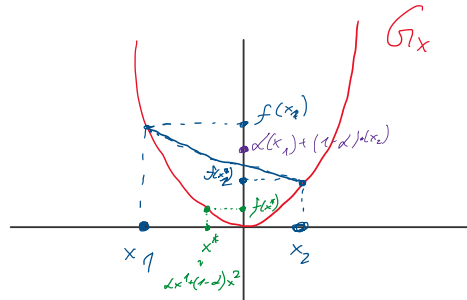
Def.

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  and  $A$  convex subset of  $\mathbb{R}^n$

Function  $f$  is called a:

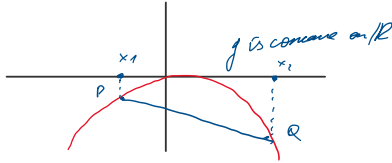
1. Concave function on  $A$  when  
 $\forall x^1, x^2 \in A, \forall a \in [0,1], f[ax^1 + (1-a)x^2] \geq af(x^1) + (1-a)f(x^2)$
2. Convex function on  $A$  when  
 $\forall x^1, x^2 \in A, \forall a \in [0,1], f[ax^1 + (1-a)x^2] \leq af(x^1) + (1-a)f(x^2)$

Eg.  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2, A = \mathbb{R}(\text{Convex})$



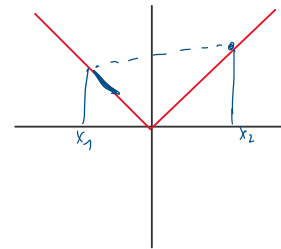
NB.  $f$  is convex if the line segment which joins any pair of points  $(x_1, f(x_1))$  and  $(x_2, f(x_2))$  of its graph lies above or on  $G_f$

$g: \mathbb{R} \rightarrow \mathbb{R}, g(x) = -x^2$



Eg.  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = |x|, A = \mathbb{R}(\text{convex})$

Given  $x_1, x_2$  in  $\mathbb{R}$  and  $a \in [0, 1]$   
 $f[ax_1 + (1-a)x_2] = [ax_1 + (1-a)x_2] \leq |ax_1| + |(1-a)x_2| = a|x_1| + (1-a)|x_2|$   
 $= af(x_1) + (1-a)f(x_2)$   
 Hence,  $f$  is convex on  $\mathbb{R}$



## Separable functions

Let  $A = A_1 \times A_2 \times \dots \times A_n \in \mathbb{R}^n, f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$   
 is said to be **separable** function if  $n$  scalar functions

$g_i: A_i \subseteq \mathbb{R} \rightarrow \mathbb{R}$   
 exist such that

$$\forall x \in A, f(x) = \sum_{i=1}^n g_i(x_i) = g_1(x_1) + g_2(x_2) + \dots + g_n(x_n)$$

Eg.  $f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x) = x_1^2 + x_2^2$   
 $g_1: \mathbb{R} \rightarrow \mathbb{R}, g_1(x_1) = x_1^2$   
 $g_2: \mathbb{R} \rightarrow \mathbb{R}, g_2(x_2) = x_2^2$

$$f(x) = \sum_{i=1}^2 g_i(x_i) - \text{is convex and separable}$$

Eg.  $g: \mathbb{R}^3 \rightarrow \mathbb{R}, g(x) = x_1^2 + x_2^2 + e^{x_3}$

$h: \mathbb{R}^3 \rightarrow \mathbb{R}, h(x) = x_1 * x_3 + x_2 * e^{x_1 x_2} - \text{non-separable}$

Eg.  $f: A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}, f(x) = \ln x_1 + \ln x_2, A = \{x \in \mathbb{R}, x_1 > 0 \text{ and } x_2 > 0\}$   
 $f$  is concave

NB.  $f, g$  convex (concave) functions on  $A \subseteq \mathbb{R}^n$   
 a.  $f + g$  convex (concave) on  $A$   
 b.  $a * f$  convex (concave) on  $A, \forall a \geq 0$

NB.  $f: \mathbb{R}^n \rightarrow \mathbb{R}, f(x) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n + b$   
 $a_1, a_2, \dots, a_n, b \in \mathbb{R}$   
 $f$  is said to be an **affine** function on  $\mathbb{R}^n$   
 $f(x) = \sum_{s=1}^n a_s x_s + b$  or  $f(x) = (a * x) + b, a \in \mathbb{R}^n$   
 $f$  is at the same time **concave** and **convex**

Eg.  $f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x) = 2x_1 - 2x_2 + 10$

## Elementary functions

Eg.  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \frac{1}{x^2} = x^{-2}$

Properties:  
 -  $f$  isn't monotone on  $\mathbb{R} - \{0\}$   
 -  $f$  isn't bounded  
 -  $f$  isn't convex

$A = (-\infty, 0) \cup (0, +\infty)$   
 not interval  $\Rightarrow$  not convex

# Elementary functions

## Elementary functions:

a. Constant functions:

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = k, k \in \mathbb{R}$$

b. Power functions

$$f: (0, +\infty) \rightarrow \mathbb{R}, f(x) = x^a, a \in \mathbb{R}$$

NB

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^n, n \in \mathbb{N}$$

c. Exponential functions:

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = a^x (a > 0, a \neq 1)$$

i.  $a > 1, f(x) = e^x$

ii.  $0 < a < 1, g(x) = \left(\frac{1}{e}\right)^x = e^{-x}$

d. Logarithmic functions:

$$f: (0, +\infty) \rightarrow \mathbb{R}, f(x) = \log_a x [a > 0, a \neq 1]$$

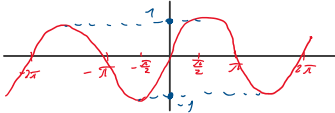
i.  $a > 1, f(x) = \ln x$

$f$  is increasing and concave, unbounded

ii.  $0 < a < 1, f(x) = \lg_{0.5} x$

e. Elementary trigonometric functions:

i. Sine function:  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \sin(x)$



$$A = \mathbb{R}, f(\mathbb{R}) = [-1, 1]$$

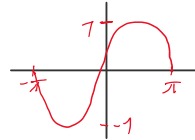
$$\forall x \in \mathbb{R}, |\sin x| \leq 1$$

$$\sin(x + 2k\pi) = \sin(x), \forall x \in \mathbb{R}, k \in \mathbb{Z}$$

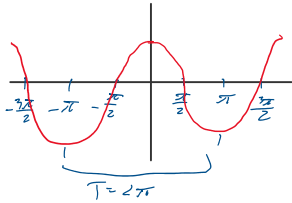
$\sin(x)$  is a periodic function with period

$$T = 2\pi$$

$$f: [-\pi, \pi] \rightarrow \mathbb{R}, f(x) = \sin(x)$$



ii. Cosine function:  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \cos(x)$



$$A = \mathbb{R}, f(\mathbb{R}) = [-1, 1]$$

$$\forall x \in \mathbb{R}, |\cos(x)| \leq 1$$

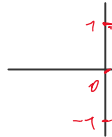
$$\cos(x + 2k\pi) = \cos(x), \forall x \in \mathbb{R}, k \in \mathbb{Z}$$

$\cos(x)$  is a periodic function with period

$$T = 2\pi$$

NB.

$$f: [0, \pi] \rightarrow \mathbb{R}$$



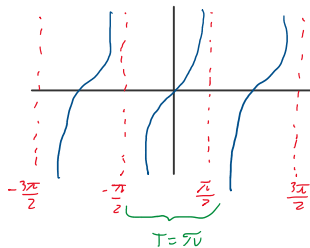
$f$ :

- strictly decreasing
- invertible

iii. Tangent function

$$f: \mathbb{R} - \left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\} \rightarrow \mathbb{R}$$

$$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$



$\tan(x)$  is:

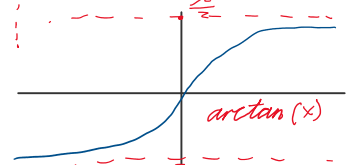
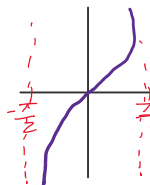
- unbounded
- periodic function with fundamental period  $T = \pi$

$$f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$f$  is:

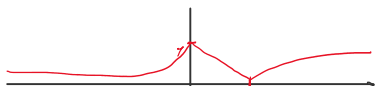
- strictly increasing, thus invertible

$$f^{-1}: \mathbb{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



Eg.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} e^{2x}, & \text{if } x < 0 \\ -x - 1, & \text{if } 0 \leq x < 1 \\ \ln x & \text{if } x \geq 1 \end{cases}$$



Properties:

- $f$  isn't monotone on  $\mathbb{R} - \{0\}$
- $f$  isn't bounded
- $f$  isn't convex

$x^-$

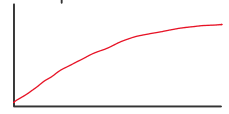


Eg.

$$f: [0, +\infty) \rightarrow \mathbb{R}, f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$f$  is:

- increasing
- concave
- not bounded
- invertible



Eg

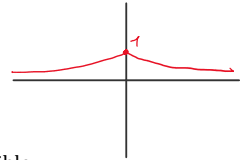
$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} e^x & \text{if } x < 0 \\ e^{-x} & \text{if } x \geq 0 \end{cases}$$

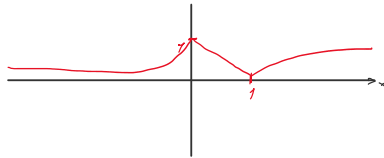
$$f(\mathbb{R}) = (1, 0]$$

$$\sup f(x) = 1$$

$$\inf f(x) = 0$$

$f$  is not monotone and not invertible





## Maximizers and minimizers

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  and  $x^*$  is said to be:

a. A **maximizer** of  $f$  on  $A$  when

$$\forall x \in A, f(x) \leq f(x^*)$$

b. A **minimizer** of  $f$  on  $A$  when

$$\forall x \in A, f(x) \geq f(x^*)$$

*→ Maximum value of  $f(x)$*

*→ Minimum value of  $f(x)$*

Eg.

$$\begin{aligned} f: \mathbb{R} \rightarrow \mathbb{R}, f(x) &= x^2 + |x| \\ \forall x \in \mathbb{R}, f(x) &= x^2 + |x| \geq 0 = f(0) \Rightarrow \forall x \in \mathbb{R}, f(x) \geq f(0) \Rightarrow \\ &\Rightarrow x = 0 \text{ is a minimizer of } f \text{ with minimum value } 0 \end{aligned}$$

Eg.

$$\begin{aligned} g: \mathbb{R}^2 \rightarrow \mathbb{R}, g(x) &= 1 - ||2x|| \\ \forall x \in \mathbb{R}^2, g(x) &= 1 - ||2x|| \leq 1 = g(0) \Rightarrow \forall x \in \mathbb{R}^2, g(x) \leq g(0) \text{ and } x = 0 \text{ is a} \\ &\text{maximizer of } g \text{ with maximum value } 1 \end{aligned}$$

Eg.

$$\begin{aligned} h: \mathbb{R} \rightarrow \mathbb{R}, h(x) &= x^2 - 2x + |x - 1| + 2 \\ &\equiv x^2 - 2x + 1 + |x - 1| + 1 = (x - 1)^2 + |x - 1| + 1 \geq 1 = h(1) \Rightarrow \\ &\Rightarrow \forall x \in \mathbb{R}, h(x) \geq h(1) \text{ and } x = 1 \text{ is a minimizer of } h \text{ with minimum value } 1 \end{aligned}$$

NB.

We introduce a notation for maximizers and minimizers

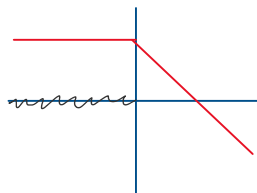
- $\operatorname{argmax}_{x \in A} f(x)$  – set of all the maximizers of  $f$  on  $A$
- $\operatorname{argmin}_{x \in A} f(x)$  – set of all the minimizers of  $f$  on  $A$
- $\max_{x \in A} f(x)$  = maximum value of  $f$  of  $A$
- $\min_{x \in A} f(x)$  = minimum value of  $f$  of  $A$

Eg.

$$\begin{aligned} f: \mathbb{R} \rightarrow \mathbb{R}, f(x) &= x^2 \\ &\bullet \operatorname{argmin}_{x \in \mathbb{R}} x^2 = \{0\} \\ &\bullet \min_{x \in \mathbb{R}} f(x) = \{0\} \end{aligned}$$

Eg.

$$\begin{aligned} g: \mathbb{R} \rightarrow \mathbb{R}, g(x) &= \begin{cases} 1, & x \leq 0 \\ -x + 1, & x > 0 \end{cases} \\ &\bullet \operatorname{argmax}_{x \in \mathbb{R}} g(x) = (-\infty, 0] \\ &\bullet \operatorname{argmin}_{x \in \mathbb{R}} f(x) = 1 \end{aligned}$$



NB

- $\operatorname{argmax}_{x \in \mathbb{R}} f(x) = \emptyset$
- $\operatorname{argmax}_{x \in \mathbb{R}} f(x) = \{x^*\}$  – strong maximizer of  $f$
- $\operatorname{argmax}_{x \in \mathbb{R}} f(x) = (a, b)$  – weak maximizer of  $f$

A function  $f: N \rightarrow R$  is called a **sequence** of real numbers

The domain of a real sequence is  $N$  or a subset of  $N$

Eg.

$$\begin{cases} f: N \rightarrow R, f(n) = 2n, n \in N \\ N = \{0, 1, \dots\} \end{cases}$$

Notation:

$\{x_n\}_{n \geq 0}$  – Sequence or real number with general term  $x_n$

Eg.  $\{n^2\}_{n \geq 0}$  or  $\{x_n\}_{n \geq 0}$  with  $x_n = n^2$  *general term of the sequence*

Eg.

$$\left\{ \frac{1}{n} \right\}_{n \geq 1} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$$

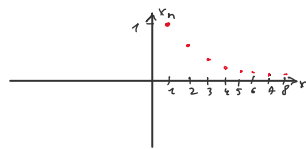
We can indicate  $\{x_n\}$  in three different ways:

- $\{x_0, x_1, x_2, \dots\}$   
Eg.  $\{1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, \dots\}$ ;  $x_n = \begin{cases} 1 & \text{if } n \leq 7 \\ 0 & \text{if } n > 7 \end{cases}$
- $x_n = \text{general term}$   
Eg.  $\{x_n\}_{n \geq 0} = x_n = n^2 + 2n$ ;  $\{0, 3, 8, 15, \dots\}$
- Using a recurrence formula  
 $\begin{cases} x_0 \in R \\ x_{n+1} = f(x_n) \end{cases}; x_0 \xrightarrow{f} x_1 \xrightarrow{f} x_2 \xrightarrow{f} x_3 \dots$   
Eg.  
 $\begin{cases} x_0 = 2 \\ x_{n+1} = 2x_n + 1, n \geq 0 \end{cases} = \{5, 11, 23, \dots\}$   
 $x_1 = 2 * x_0 + 1 = 5$   
 $x_2 = 2 * x_1 + 1 = 11$   
 $x_3 = 2 * x_3 + 1 = 24$

**Remarkable sequences:**

**1. Harmonic Sequence**

$$\left\{ \frac{1}{n} \right\}_{n \geq 1} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$$



**2. Arithmetic sequence: given  $a, r \in R$**

$$\{a + rn\}_{n \geq 0} = \{a, a + r, a + 2r, a + 3r, \dots\}$$

Arithmetic sequence with first term  $a$  and ratio  $r$ .

Recurrence formula  $\begin{cases} x_0 = a \in R \\ x_{n+1} = x_n + r \end{cases}$

Through general term:  $\{x_n\}_{n \geq 0} x_n = a + rn$

**3. Geometric sequence: given  $a, q \in R$**

$$\{a * q^{n-1}\}_{n \geq 1} = \{a, aq, aq^2, aq^3, \dots\}$$

Arithmetic sequence with first term  $a$  and ratio  $q$ .

Recurrence formula:  $\begin{cases} x_0 = a \in R \\ x_{n+1} = qx_n \end{cases}$

Through general term:  $\{x_n\}_{n \geq 1}$  with  $x_n = aq^{n-1}$

$R^\infty \equiv R^N = \text{Space of all the sequences of real numbers}$

NB.

$x \in R^\infty, x$  – sequence of real number:  $x = \{x_n\} = \{x_0, x_1, x_2, \dots\}$

NB.

$x \in R^\infty, x$  – sequence of real number:  $x = \{x_n\} = \{x_0, x_1, x_2, \dots\}$

Consider  $x, y \in R^\infty$ . We have:

1.  $(x + y)_n = x_n + y_n$
2.  $(x * y)_n = x_n * y_n$
3.  $\left(\frac{x}{y}\right)_n = \frac{x_n}{y_n}, \forall n \geq 0, y_n \neq 0$

NB.

$x_n = 0, \forall n \geq 0$  – **Null sequence**  
 $\{0\} = \{0, 0, 0, 0, \dots\}$

In  $R^N$  (Space) we have a partial order structure like  $R^n$  (space of vectors/points of  $R^n$ )

1. Weak inequality:  $x \geq y$  if  $x_n \geq y_n, \forall n \geq 1$
  2. Strict inequality:  $x > y$  if  $x \geq y$  and  $x \neq y, \forall n \geq 1$
  3. Strong inequality:  $x \gg y$  if  $x_n > y_n, \forall n \geq 1$
- $$\begin{cases} x \gg y \Rightarrow x > y \Rightarrow x \geq y \\ x, y \in R^N \end{cases}$$

A sequence  $\{x_n\}$  is said to be:

1. Bounded from above if  $\forall n \geq 1, \exists k \in R: x_n \leq k$
2. Bounded from below if  $\forall n \geq 1, \exists h \in R: x_n \geq h$
3. Bounded if  $\forall n \geq 1, \exists h, k \in R: h \leq x_n \leq k$

OR:

$$(\forall n \geq 1, \exists k > 0: |x_n| \leq k)$$

Eg.

$$\left\{\frac{1}{n^2}\right\}_{n \geq 1}, 0 < \frac{1}{n^2} \leq 1 \text{ – bounded } (h = 0, k = 1)$$

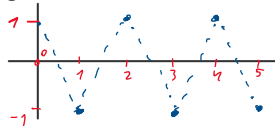
Eg.

$$\{-n^2\}_{n \geq 1}, x_n = -n^2 \leq 0$$

Bounded from above, but not bounded in general

NB.

$$\{(-1)^n\}, x_n = (-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$



It is bounded

$\{x_n\}$  is said to be:

1. **Increasing** if  $\forall n \geq 1, x_n \leq x_{n+1}$   
**Strictly increasing** if  $\forall n \geq 1, x_n < x_{n+1}$
2. **Decreasing** if  $\forall n \geq 1, x_n \geq x_{n+1}$   
**Strictly decreasing** if  $\forall n \geq 1, x_n > x_{n+1}$
3. **Constant** if  $\{x_n\}$  is increasing and decreasing at the same time, i.e.  $x_n = k, \forall n \geq 1$

NB.

$$\forall n \geq 1, x_n \leq x_{n+1} \Rightarrow x_{n+1} - x_n \geq 0$$

Eg.

$$\begin{aligned} \{x_n\}, x_n &= n^2 \\ x_{n+1} - x_n &= (n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1 \geq 0 \\ \text{Hence, } \{x_n\} &\text{ is increasing} \end{aligned}$$

Eg.

$$\begin{aligned} \left\{\frac{1}{n+1}\right\}_{n \geq 0} &= \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\} \\ \forall n \geq 0, x_{n+1} - x_n &= \frac{1}{(n+1)+1} - \frac{1}{n+1} = \frac{n+1-n-2}{(n+2)(n+1)} = -\frac{2}{(n+2)(n+1)} \leq 0 \end{aligned}$$

Hence,  $\{x_n\}$  is decreasing

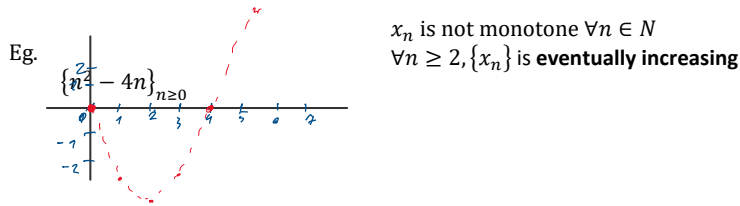
Eg.

Positive increasing sequence  $x_n > 0, \forall n \geq 1 \Leftrightarrow$

1.  $\forall n \geq 1, x_n \leq x_{n+1}$

2.  $\forall n \geq 1, \frac{x_n}{x_{n+1}} \leq 1$

A sequence such that  $\forall n \geq n^*$  a property  $p$  holds is called **eventually sequence**



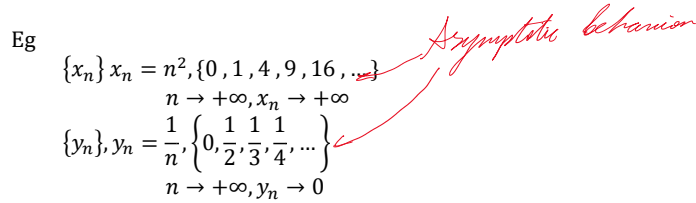
Eg

$\{x_n\}$  with  $x_n = n^2 - 4n$  and  $n \geq 0$

$n$	0	1	2	3	4	5	6
$x_n$	0	-3	-4	-3	0	5	12

- $\{x_n\}$  is not monotone ( $\forall n \geq 0$ )
- $\{x_n\}$  is eventually/definitively increasing ( $\exists n^* = 2: \forall n \geq 2, x_n \leq x_{n+1}$ )
- $\{x_n\}$  is eventually/definitively increasing and strictly positive ( $\exists n' = 5: \forall n \geq 5, x_n \leq x_{n+1}$  and  $x_n > 0$ )

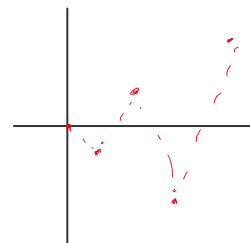
## Limits and asymptotic behaviors



We can consider 3 different asymptotic behaviors:

- $\{x_n\}$  **converges** to real number  $L (L \in \mathbb{R})$   
 Eg:  $y_n = \frac{1}{n}$  converges to  $0 \in \mathbb{R}$
- $\{x_n\}$  **diverges** to either  $-\infty$  or  $+\infty$
- $\{x_n\}$  **oscillates**  
 Eg.  $\{x_n\}_{n \geq 0}, n = (-1)^n * n^2, \{0, -1, 4, -9, 16, \dots\}$

*Regular sequence*  
*Irregular sequence*



### 1. Convergent sequences

**Definition**  
 $\{x_n\}$  real sequence  $\{x_n\}$  **converges** to  $L \in \mathbb{R}$   
 $(x_n \rightarrow L \text{ or } \lim_{n \rightarrow +\infty} x_n = L)$  if  
 $\forall \varepsilon > 0, \exists n_\varepsilon \geq 1: n \geq n_\varepsilon \Rightarrow |x_n - L| < \varepsilon$   
 and  $L$  is called the limit of  $\{x_n\}$



NB  
 $d(x_n, L) = |x_n - L|$  therefore  $x_n \rightarrow L, (n \rightarrow +\infty)$  if  
 $\forall \varepsilon > 0, \forall n_\varepsilon \geq 1: n \geq n_\varepsilon \Rightarrow d(x_n, L) < \varepsilon$

Eg. Show (using the definition) that  $\lim_{n \rightarrow +\infty} \frac{1}{n} = 0$

$\forall \varepsilon > 0, \exists n_\varepsilon \geq 1: n \geq n_\varepsilon \Rightarrow \left| \frac{1}{n} - 0 \right| < \varepsilon$

- Suppose the existence of  $x_\varepsilon \geq 1$  and  $n \geq n_\varepsilon$

$$\left| \frac{1}{n} - 0 \right| < \varepsilon \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon}, n_\varepsilon = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$$

2. Taking  $n_\varepsilon = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$  and assume that  $n \geq n_\varepsilon$  we have

$$\left| \frac{1}{n} - 0 \right| = \left| \frac{1}{n} \right| = \frac{1}{n} \leq \frac{1}{n_\varepsilon} = \frac{1}{\left\lceil \frac{1}{\varepsilon} \right\rceil + 1} < \frac{1}{\frac{1}{\varepsilon} - 1 + 1} = \frac{1}{\varepsilon} = \varepsilon$$

Therefore,  $\left| \frac{1}{n} - 0 \right| < \varepsilon$

NB.

A neighborhood of  $L \in R$  with radius  $\varepsilon > 0$  is given by  $B_\varepsilon(n) = \{x \in R: d(x_n, L) < \varepsilon\}$   
 $= \{x \in R: |x_n - L| < \varepsilon\} = \{L - \varepsilon, L + \varepsilon\}$

Def

$\{x_n\}$  real sequence  $\{x_n\}$  converges to  $L \in R$  if  $\forall B_\varepsilon(L), \exists n_\varepsilon \geq 1: n \geq n_\varepsilon \Rightarrow x_n \in B_\varepsilon(L)$



Proof:

1. Suppose the existence of  $n_\varepsilon \geq$  and  $n \geq n_\varepsilon$

$$\frac{2}{n} \in B_\varepsilon(0) \Rightarrow \frac{2}{n} \in (-\varepsilon, \varepsilon) \Rightarrow -\varepsilon < \frac{2}{n} < \varepsilon \Rightarrow \begin{cases} \frac{2}{n} > -\varepsilon \\ \frac{2}{n} < -\varepsilon \end{cases} \Rightarrow \frac{2}{n} < \varepsilon \Rightarrow \frac{1}{n} < \frac{\varepsilon}{2} \Rightarrow n > \frac{2}{\varepsilon}$$

$$n_\varepsilon = \left\lceil \frac{2}{\varepsilon} \right\rceil + 1$$

2. Taking  $n_\varepsilon = \left\lceil \frac{2}{\varepsilon} \right\rceil + 1$  and assume that  $n \geq n_\varepsilon$

NB.

We have to show that  $\frac{2}{n} \in B_\varepsilon(0)$

$$0 < \frac{2}{n} \leq \frac{2}{n_\varepsilon} = \frac{2}{\left\lceil \frac{2}{\varepsilon} \right\rceil + 1} < \frac{2}{\frac{2}{\varepsilon} - 1 + 1} = \frac{2}{\frac{2}{\varepsilon}} = \varepsilon$$

$$\Rightarrow 0 < \frac{2}{n} < \varepsilon \Rightarrow -\varepsilon < 0 < \frac{2}{n} < \varepsilon \Rightarrow -\varepsilon < \frac{2}{n} < \varepsilon \Rightarrow \frac{2}{n} \in B_\varepsilon(0)$$

NB

$$x \in R \text{ positive } \frac{1}{[x]} < \frac{1}{x-1}$$

Any sequence converging to 0 ( $x_n \rightarrow 0$ ) is called an infinitesimal or null sequence ( $\neq$  zero sequence  $\{0,0,0, \dots\}$ )

$$x_n = \frac{1}{2n} \rightarrow 0, n \rightarrow +\infty, \left\{ \frac{1}{2n} \right\} \text{ infinitesimal sequence}$$

## Proposition 306

$\{x_n\}$  real sequence

$\{x_n\}$  converges to  $L \in R \Leftrightarrow d(x_n, L) \rightarrow 0$

Proof

( $\Leftarrow$ )

Assume that  $d(x_n, L) \rightarrow 0$  as  $n \rightarrow +\infty$ , i.e.  $\forall \varepsilon > 0, \exists n_\varepsilon \geq 1: n \geq n_\varepsilon \Rightarrow d(x_n, L) < \varepsilon$ . We suppose that  $\{x_n\}$  is not convergent to  $L \in R$ , i.e.

$$\exists \varepsilon > 0: \forall n \geq n_\varepsilon: n \geq n_\varepsilon \text{ and } |x_n - L| \geq \varepsilon$$

From the premise,  $d(x_n, L) < \varepsilon$  is equivalent to  $|x_n - L| < \varepsilon$ .

Our supposition is false and  $\{x_n\}$  converges to  $L \in R$

( $\Rightarrow$ )

Assume that  $x_n$  converges to  $L \in R$ , i.e., by definition,  $\forall \varepsilon > 0, \exists n_\varepsilon \geq 1: n \geq n_\varepsilon \Rightarrow$

$$|x_n - L| < \varepsilon$$

Given  $\varepsilon > 0$ , we have:

$$d(x_n, L) = |x_n - L| < \varepsilon \Rightarrow d(x_n, L) < \varepsilon$$

Therefore  $\forall \varepsilon > 0, \exists n_\varepsilon \geq 1: n \geq n_\varepsilon \Rightarrow d(x_n, L) < \varepsilon$ , therefore  $d(x_n, L) \rightarrow 0$

Eg.

Show that  $\lim_{n \rightarrow +\infty} \left[ \frac{(-1)^n}{n^2} + 2 \right] = 2$

$$d(x_n, 2) = |x_n - 2| = \left| \frac{(-1)^n}{n^2} + 2 - 2 \right| = \left| \frac{(-1)^n}{n^2} \right| = \frac{|(-1)^n|}{n^2} = \frac{1}{n^2} \rightarrow 0$$

NB.

- $x_n \rightarrow L^+$  or  $\lim_{n \rightarrow +\infty} x_n = L^+$   
if  $x_n \rightarrow L$  and eventually  $x_n \geq L$

$\{x_n\}$  is decreasing we can write  $x_n \downarrow L$   
Eg.

$$\{x_n\}x_n = \begin{cases} -1, n \leq 20 \\ \frac{1}{n^2}, n > 20 \end{cases}$$

$$x_n \rightarrow 0 \text{ and } x_n \geq 0, \forall x \geq 2$$

Therefore,  $x_n \rightarrow 0^+$  or  $\lim_{n \rightarrow +\infty} x_n = 0^+$  or  $(x_n \downarrow 0)$

- $x_n \rightarrow L^-$  or  $\lim_{n \rightarrow +\infty} x_n = L^-$   
if  $x_n \rightarrow L$  and eventually  $x_n \leq L$   
 $\{x_n\}$  is increasing we can write  $(x_n \uparrow L)$

Divergence sequences

Def.  $\{x_n\}$  real sequence.  $\{x_n\}$  **diverges** to  $+\infty$  ( $x_n \rightarrow +\infty$  or  $\lim_{n \rightarrow +\infty} x_n = +\infty$ ) if  
 $\forall k \in \mathbb{R}, \exists n_k \geq 1: n \geq n_k \Rightarrow x_n > k$

Eg.

$$\{x^3\} = \{1, 8, 27, 64, \dots\}$$

$$n^3 \rightarrow +\infty$$

$$\forall k \in \mathbb{R}, \exists n_k \geq 1: n \geq n_k \Rightarrow n^3 > k$$

Eg.

$$\{x_n\} \text{ with } x_n = n^2, \lim_{n \rightarrow +\infty} n^2 = +\infty$$

$$\left\{ \begin{array}{l} k \in \mathbb{R}, \exists n_k \geq 1: n \geq n_k \Rightarrow n^2 > k \\ \forall k > 0, \exists n_k \geq 1: n \geq n_k \Rightarrow n^2 > k \end{array} \right.$$

$$\left\{ \begin{array}{l} \forall k > 0, \exists n_k \geq 1: n \geq n_k \Rightarrow n^2 > k \end{array} \right.$$

Let  $k$  be a positive real number.

- Suppose the existence of  $n_k \geq 1$  and  $n \geq n_k$

$$n^2 > k \Rightarrow n > \sqrt{k}, n_k = \lceil \sqrt{k} \rceil + 1$$

- Taking  $n_k = \lceil \sqrt{k} \rceil + 1$  and assume that  $n \geq n_k$

$$n^2 \geq (n_k)^2 \Rightarrow n^2 \geq (\lceil \sqrt{k} \rceil + 1)^2 > (\lceil \sqrt{k} \rceil + 1 - 1)^2 = \lceil \sqrt{k} \rceil^2 = k$$

Therefore,  $n^2 > k$

Def

$\{x_n\}$  real sequence

$\{x_n\}$  diverges to  $-\infty$  ( $x_n \rightarrow -\infty$  or  $\lim_{n \rightarrow +\infty} x_n = -\infty$ ) if  
 $\forall k \in \mathbb{R}, \exists n_k \geq 1: n \geq n_k \Rightarrow x_n < k$

Eg

$$\lim_{n \rightarrow +\infty} (-n^2) = -\infty$$

$$\forall k < 0, \exists n_k \geq 1: n \geq n_k \Rightarrow -n^2 < k$$

Any sequence diverging to either  $+\infty$  or  $-\infty$  is called an **infinite sequence**

Eg.

$$\{n^2\}, \left\{ \frac{1}{n^2} \right\}$$

## Proposition 313

$$\{x_n\} \text{ diverges to } +\infty \Leftrightarrow \left\{ \frac{1}{x_n} \right\} \text{ converges to } 0$$

**Proof:**

$\{x_n\}$  is an eventually positive sequence  
( $\Leftarrow$ )

Assume that  $\left\{ \frac{1}{x_n} \right\}$  converges to 0 ( $\forall \varepsilon > 0, \exists n_\varepsilon \geq 1, n \geq n_\varepsilon \Rightarrow \frac{1}{x_n} < \varepsilon$ ). Taking

$\varepsilon = \frac{1}{k} > 0$ , therefore a position  $n_{\frac{1}{k}}$  exists such that

$$0 < \frac{1}{x_n} < \frac{1}{k} \Rightarrow x_n > k, \forall n \geq n_k$$

and  $\{x_n\}$  diverges to  $+\infty$

( $\Rightarrow$ )

Assume that  $\{x_n\}$  diverges to  $+\infty$  ( $\forall k > 0, \exists n_k \geq 1: n \geq n_k \Rightarrow x_n > k$ )

Taking  $k = \frac{1}{\varepsilon} > 0$ , a position  $n_{\frac{1}{\varepsilon}} \geq 1$  exists such that  $x_n > \frac{1}{\varepsilon}$  for any  $n \geq n_{\frac{1}{\varepsilon}}$ .  
 Therefore  $\frac{1}{x_n} < \varepsilon$  for any  $n \geq n_{\frac{1}{\varepsilon}}$  and  $\left\{\frac{1}{x_n}\right\}$  converges to 0.

Eg.

$$\{x_n\} \text{ with } x_n = \frac{1}{n^2}, \frac{1}{n^2} \rightarrow 0 \text{ as } n \rightarrow +\infty \text{ OR } \lim_{(n \rightarrow +\infty)} \frac{1}{n^2} = 0$$

$$\{y_n\} \text{ with } y_n = \begin{cases} -1, n \leq 100, \\ n^2 > 100 \end{cases}$$

$\{y_n\}$  isn't monotone, but is eventually monotone and positive

$$\forall n^* \geq 101, \exists x^* \geq 101, y_n > 0 \text{ and } y_n < y_{n+1}$$

$$\lim_{(n \rightarrow +\infty)} y_n = \lim_{(n \rightarrow +\infty)} n^2 = +\infty$$

$$\{z_n\} \text{ with } z_n = \begin{cases} n^2, n \text{ even} & \lim_{(n \rightarrow +\infty)} n^2 = +\infty \\ \frac{1}{n^2}, n \text{ odd} & \lim_{(n \rightarrow +\infty)} \frac{1}{n^2} = 0 \end{cases}$$

$\nexists \lim_{(n \rightarrow +\infty)} z_n$  - no limit for an entire function

### Theorem 318 (Uniqueness of the limit)

$\{x_n\}$  real sequence.  $\{x_n\}$  converges to at most one limit  $L \in \bar{R}$

Proof

1.  $L \in R$ ;
2.  $L = +\infty (L = -\infty)$

i. Suppose that  $L \in R$

Proof by contradiction. Suppose the existence of two limits  $L_1 \in R$  and  $L_2 \in R, L_1 \neq L_2$

Assume that  $L_1 < L_2$ . We have

$$0 \leq |L_2 - L_1| = |(L_2 - x_n) + (x_n - L_1)| \leq |L_2 - x_n| + |x_n - L_1| \\ = |x_n - L_2| + |x_n - L_1| < \varepsilon + \varepsilon = 2\varepsilon$$

Therefore, we get  $0 \leq |L_2 - L_1| < 2\varepsilon, \forall \varepsilon > 0$  and we have  $L_2 = L_1$

The limit of  $\{x_n\}$ , if it exists, is unique

ii.  $L = \pm\infty$ .

a. Proof by contradiction. We suppose the existence of two limits  $L_1 \in R$  and  $L_2 = +\infty$

By definition, for any  $\varepsilon > 0$  and for any  $k > 0$  two positions  $n_\varepsilon$  and  $n_k$  exist such that

$$L_1 - \varepsilon < x_n < L_1 + \varepsilon, \forall n \geq n_\varepsilon \text{ and } x_n > k, \forall n \geq n_k$$

Let  $n^* = \max\{n_\varepsilon, n_k\}$ , therefore  $\forall n \geq n^*$  we have

$$L_1 - \varepsilon < x_n < L_1 + \varepsilon \text{ and } x_n > k$$

Taking  $k = L_1 + \varepsilon$  we have a contradiction to our startin supposition is false. The limit of  $\{x_n\}$ , if it exists, is unique

b. Do by myself a proof for  $L_2 = -\infty$

Eg.

$$y_n = \begin{cases} \frac{1}{n^2}, n \text{ even} \\ \frac{1}{2n}, n \text{ odd} \end{cases} \lim_{n \rightarrow +\infty} \frac{1}{n^2} = 0; \lim_{n \rightarrow +\infty} \frac{1}{2n} = 0, \exists \lim_{n \rightarrow +\infty} y_n = 0, \{y_n\} \text{ regular sequence}$$

### Theorem 319 (Permanence of sign)

Let  $\{x_n\}$  be a sequence such that  $x_n \rightarrow L \neq 0$ . Eventually (Definitively)  $\{x_n\}$  has the same sign of  $L$ , that is  $x_n * L > 0$

Proof

Assume that  $L > 0 (L < 0)$ . By definition of limit, for any  $\varepsilon > 0$  a position  $n^* \geq 1$  exists such that

$|x_n - L| < \varepsilon$  for any  $n \geq n^*$ . Taking  $\varepsilon \in (0, L)$ , i.e.  $0 < \varepsilon < L$ , we obtain  $L - \varepsilon > 0$  and

$$0 < L - \varepsilon < x_n (x_n > 0)$$

Therefore,  $x_n > 0$  for any  $n \geq n^*$

Prove for negatives

NB

$$\{x_n\}, x_n = \frac{1}{n} \rightarrow 0 = L,$$

$$x_n = \frac{1}{n} > 0, \forall n \geq 1$$

### Proposition 320

$\{x_n\}, \{y_n\}$  real sequences such that  $x_n \rightarrow L \in \bar{R}, y_n \rightarrow H \in \bar{R}$ . If eventually  $x_n \geq y_n$ , then  $L \geq H$ .



### Proposition 320

$\{x_n\}, \{y_n\}$  real sequences such that  $x_n \rightarrow L \in \bar{R}, y_n \rightarrow H \in \bar{R}$ . If eventually  $x_n \geq y_n$ , then  $L \geq H$ .  
!!! converse of the proposition is false



### Proposition 321

$\{x_n\}, \{y_n\}$  real sequences such that  $x_n \rightarrow L \in \bar{R}, y_n \rightarrow H \in \bar{R}$ . If  $L > H$ , then eventually  $x_n > y_n$

Proof

Let  $\{x_n\}, \{y_n\}$  be sequences such that  $x_n \rightarrow L \in R$  and  $y_n \rightarrow H \in R$ , therefore, from the definition of limit, we have

$$\forall \varepsilon > 0, \exists n'_\varepsilon \geq 1: n \geq n'_\varepsilon \Rightarrow |x_n - L| < \varepsilon$$

$$\forall \varepsilon > 0, \exists n''_\varepsilon \geq 1: n \geq n''_\varepsilon \Rightarrow |y_n - H| < \varepsilon$$

Since  $L > H$  ( $L - H > 0$ ) we can take  $0 < \varepsilon < \frac{L-H}{2}$  and we have  $H + \varepsilon < L - \varepsilon$ .

Therefore

$$x_n > L - \varepsilon > H + \varepsilon > y_n \Rightarrow x_n > y_n$$

For any  $n \geq n^* = \max\{n'_\varepsilon, n''_\varepsilon\}$

## Monotonicity and convergence

Eg.

- $x_n = \frac{1}{n+1}$  - is decreasing and bounded from below  $\Rightarrow$  is convergent  

$$x_{n+1} - x_n = \frac{1}{(n+1)+1} - \frac{1}{n+1} = \frac{1}{n+2} - \frac{1}{n+1} = \frac{n+1-n-2}{(n+2)(n+1)} = \frac{-1}{(n+2)(n+1)}$$

$$\frac{1}{n+1} \rightarrow 0 = -\frac{1}{(n+2)(n+1)} < 0, \forall n \geq 1, \{x_n\}$$
 is decreasing  
 NB.  $\begin{cases} \lim_{n \rightarrow +\infty} \frac{1}{n+1} = 0 \\ \forall \varepsilon > 0, \exists n_\varepsilon \geq 0: n \geq n_\varepsilon \Rightarrow \left| \frac{1}{n+1} - 0 \right| < \varepsilon \end{cases}$
- $y_n = 1 - \frac{1}{n^2}$  - is increasing and bounded from above  $\Rightarrow$  is convergent  

$$\lim_{n \rightarrow +\infty} \left[ 1 - \frac{1}{n^2} \right] = 1$$
- $z_n = n^2$  - is increasing and unbounded from above  $\Rightarrow$  diverges to  $+\infty$
- $t_n = -n^3$  - is decreasing and unbounded from below  $\Rightarrow$  diverges to  $-\infty$

### Proposition 322

Each convergent sequence is bounded

Another way to write it

Let  $\{x_n\}$  be a real sequence. If  $\{x_n\}$  is convergent, then  $\{x_n\}$  is bounded

Proof

Assume that  $\{x_n\}$  is convergent to  $L \in R$ , i.e.  $x_n \rightarrow L$  as  $n \rightarrow +\infty$ . Taking  $\varepsilon = 1$ , a position  $n_1 \geq 1$  exists such that  $x_n \in B_1(L)$  for any  $n \geq n_1$ . Let  $M \in R$  be such that

$$M > \max\{1, d(x_1, L), d(x_2, L), \dots, d(x_{n_1}, L)\}$$

therefore  $M > d(x_n, L)$  for any  $n \geq 1$ . Since  $|x_n - L| \leq M, \forall n \geq 1$ , we have

$$-M < x_n - L < M \Rightarrow L - M < x_n < L + M$$

therefore  $\{x_n\}$  is bounded

Eg.

$$x_n = \frac{1}{n^2} \rightarrow 0 \Rightarrow \{x_n\} \text{ is bounded } 0 < \frac{1}{n^2} \leq 1, \forall n \geq 1$$

NB - converse is false

$$x_n = (-1)^n = \begin{cases} -1, n \text{ odd} \\ 1, n \text{ even} \end{cases}$$

$$-1 \leq x_n \leq 1 \Rightarrow \{x_n\} \text{ is bounded}$$

$$x_n = -1 \rightarrow -1; x_n \rightarrow 1$$

### Theorem 323

Each monotone sequence is regular

- $\{x_n\}$  bounded  $\Rightarrow \{x_n\}$  converges
- $\{x_n\}$  increasing and unbounded  $\Rightarrow \{x_n\}$  diverges to  $+\infty$
- $\{x_n\}$  decreasing and unbounded  $\Rightarrow \{x_n\}$  diverges to  $-\infty$

Proof

Assume that  $\{x_n\}$  is increasing. Therefore  $\{x_n\}$  can be either bounded from above or

unbounded from above  $x_1 \leq x_n \forall n \geq 1$

**1.  $\{x_n\}$  bounded**

Let  $E$  be the image set of  $\{x_n\}$

$E$  is a bounded subset of  $R$ , therefore  $\sup E$  exists. Set  $\sup E = L$ .

Given  $\varepsilon > 0$ , since  $L$  is the supremum of  $E$  we have

i.  $\forall n \geq 1, x_n \leq L$

ii.  $\exists x_{n_\varepsilon} \in E: x_{n_\varepsilon} > L - \varepsilon, \forall n \geq n_\varepsilon$  ( $L$  is the least upper bound)

$\{x_n\}$  is increasing, we have

$$L \geq x_n \geq x_{n_\varepsilon} > L - \varepsilon, \forall n \geq n_\varepsilon$$

therefore  $x_n \in B_\varepsilon(L)$  for any  $n \geq n_\varepsilon$

**2.  $\{x_n\}$  unbounded**

For any  $k > 0$  an element  $x_{n_k}$  exists such that  $x_{n_k} > k$ . Since  $\{x_n\}$  is increasing, we have

$$x_n \geq x_{n_k} > k, \forall n \geq n_k$$

therefore  $\{x_n\}$  diverges to  $+\infty$

**3.  $\{x_n\}$  decreasing (bounded and unbounded) - consider at home**

Eg.

$$\{x_n\} = \frac{1}{n} + \frac{1}{n^2} + 2 \Rightarrow \{x_n\} \text{ converges}$$

$$|x_n| = \left| \frac{1}{n} + \frac{1}{n^2} + 2 \right| \leq \left| \frac{1}{n} \right| + \left| \frac{1}{n^2} \right| + 2$$

$$\exists k \in R: |x_n| \leq k \leq 1 + 1 + 2 = 4$$

**Corollary**

$\{x_n\}$  monotone is convergent  $\Leftrightarrow \{x_n\}$  is bounded

## Algebra of limits

### Proposition 333

$\{x_n\}, \{y_n\}$  real sequences such that  $x_n \rightarrow L \in \bar{R}, y_n \rightarrow H \in \bar{R}$ . Then:

a.  $x_n + y_n \rightarrow L + H$  provided that  $L + H$  is not  $+\infty - \infty$  or  $-\infty + \infty$

b.  $x_n * y_n \rightarrow L * H$  provided that  $LH$  is not  $0 * (\pm\infty)$  or  $(\pm\infty) * 0$

c.  $\frac{x_n}{y_n} \rightarrow \frac{L}{H}$  provided that  $y_n \neq 0$  eventually and  $\frac{L}{H}$  is not  $0/0$  or  $\infty/\infty$

Proof

$x_n \rightarrow L \in \bar{R}, y_n \rightarrow H \in \bar{R}$ . By definition of limit,  $\forall \varepsilon > 0$  two positions  $n_\varepsilon^1, n_\varepsilon^2 \geq 1$  exist such that

$$\begin{cases} L - \varepsilon < x_n < L + \varepsilon, \forall n \geq n_\varepsilon^1 \\ H - \varepsilon < y_n < H + \varepsilon, \forall n \geq n_\varepsilon^2 \end{cases}$$

Let  $n^*$  be a position such that  $n^* = \max\{n_\varepsilon^1, n_\varepsilon^2\}$

For any  $n \geq n^*$  we have

$$(L - \varepsilon) + (H - \varepsilon) < x_n + y_n < (L + \varepsilon) + (H + \varepsilon) \Rightarrow L + H - 2\varepsilon < x_n + y_n$$

$$< (L + H) + 2\varepsilon$$

$$\Rightarrow |(x_n + y_n) - (L + H)| < 2\varepsilon$$

Therefore  $x_n + y_n \rightarrow L + H$

Study the book for other cases

### Proposition 335

$x_n \rightarrow L \in \bar{R}$  with  $L \neq 0$  and  $y_n \rightarrow 0$ . We have

$\lim_{n \rightarrow +\infty} \frac{x_n}{y_n}$  exists  $\Leftrightarrow \{y_n\}$  definitively has constant sign

1. If  $L > 0$  and  $y_n \rightarrow 0^+$  or  $L < 0$  and  $y_n \rightarrow 0^-$  then

$$\lim_{n \rightarrow +\infty} \frac{x_n}{y_n} = +\infty$$

2. If  $L > 0$  and  $y_n \rightarrow 0^-$  or  $L < 0$  and  $y_n \rightarrow 0^+$  then

$$\lim_{n \rightarrow +\infty} \frac{x_n}{y_n} = -\infty$$

Eg.

$$x_n = 1 + \frac{2}{n^3} \rightarrow 1, y_n = \frac{1}{n^3} \rightarrow 0, \frac{x_n}{y_n} \rightarrow +\infty$$

### Proposition 337

If we don't consider the following indeterminate forms  $1^{+\infty}, 1^{-\infty}, 0^0$  and  $(+\infty)^0$  we have

1.  $\lim_{n \rightarrow +\infty} x_n^a = \left( \lim_{n \rightarrow +\infty} x_n \right)^a, a \in \mathbb{R} \text{ and } x_n > 0$
2.  $\lim_{n \rightarrow +\infty} a^{x_n} = a^{\lim_{n \rightarrow +\infty} x_n}, a > 0$
3.  $\lim_{n \rightarrow +\infty} \log_b x_n = \log_b \left( \lim_{n \rightarrow +\infty} x_n \right), b > 0, b \neq 1$

Eg.

$$\lim_{n \rightarrow +\infty} n^3 = \left( \lim_{n \rightarrow +\infty} n \right)^3 = +\infty$$

Remarkable limits:

1. Power sequences:  $x_n = n^a, n \in \mathbb{N}, a \in \mathbb{R}$

$$\lim_{n \rightarrow +\infty} n^a = \begin{cases} +\infty & a > 0 \\ 1 & a = 0 \\ 0 & a < 0 \end{cases}$$

Eg.

$$\lim_{n \rightarrow +\infty} \frac{1}{\sqrt{n}} = \lim_{n \rightarrow +\infty} n^{-0.5} = 0: \left\{ \frac{1}{\sqrt{n}} \right\} \text{ infinitesimal sequence}$$

Eg.

$$\lim_{n \rightarrow +\infty} n^{a^2-1}, a \in \mathbb{R} \quad \text{Irregular sequence}$$

- $a^2 - 1 > 0 \Rightarrow a^2 > 1 \Rightarrow a < -1 \text{ or } a > 1, n^{(a^2-1)} \rightarrow +\infty$
- $a^2 - 1 = 0 \Rightarrow a^2 = 1 \Rightarrow a = \pm 1, n^{(a^2-1)} \rightarrow 1$
- $a^2 - 1 < 0 \Rightarrow a^2 < 1 \Rightarrow -1 < a < 1, n^{(a^2-1)} \rightarrow 0$

2. Exponential (geometric) sequence:  $x_n = a^n, a \in \mathbb{R}$

$$\lim_{n \rightarrow +\infty} a^n = \begin{cases} +\infty & a > 1 \\ 1 & a = 1 \\ 0 & -1 < a < 1 \\ \nexists & a \leq -1 \end{cases} \quad \text{Regular sequence}$$

Eg.

$$x_n = 3^n \rightarrow +\infty$$

$$x_n = \left( -\frac{1}{2} \right)^n \rightarrow 0$$

$$x_n = (-1)^n, \nexists \lim_{n \rightarrow +\infty} x^n$$

3. Logarithmic sequences:  $x_n = \log_b x, b > 0, b \neq 1$

$$\lim_{n \rightarrow +\infty} \log_b n = \begin{cases} +\infty, b > 1 \\ -\infty, 0 < b < 1 \end{cases}$$

NB

$\{x_n\}, x_n = n!, \forall n \in \mathbb{N}, n! = 1 * 2 * \dots * n$  and  $0! = 1$

1.  $\{x_n\}$  is increasing,  $n! = \{1, 1, 2, 6, 24, 120, \dots\}$
2.  $\{x_n\}$  unbounded from above
3.  $\{x_n\}$  diverges to  $+\infty$

$\left\{ \frac{1}{n!} \right\}$  – Infinitesimal sequence

1. Monotone decreasing
2. bounded from below

Eg.

$\{x_n\}, x_n = n^3 - n^2 \rightarrow \text{indef. form } +\infty - \infty$

$$x_n = n^3 - n^2 = n^3 \left[ 1 - \frac{n^2}{n^3} \right] \rightarrow +\infty$$

Hence,  $x_n \rightarrow +\infty$

Eg.

$$\{x_n\} = \sqrt{n^2 + n} - n \rightarrow \text{Ind. form } +\infty - \infty$$

NB:  $y_n = \sqrt{n^2 + n} \rightarrow +\infty$

$$\frac{n}{n \left[ \sqrt{1 + \frac{1}{n}} + 1 \right]} \frac{(\sqrt{n^2 + n} - n)(\sqrt{n^2 + n} + n)}{\sqrt{n^2 + n} + n} = \frac{(\sqrt{n^2 + n})^2 - n^2}{\sqrt{n^2 + n} + n} = \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n}$$

$$= \frac{n}{n \sqrt{1 + \frac{1}{n}} + n} = \frac{1}{\left[ \sqrt{1 + \frac{1}{n}} + 1 \right]} \rightarrow \frac{1}{2}$$

$$\text{Therefore, } \{x_n\} = \sqrt{n^2 + n} - n \rightarrow \frac{1}{2}$$

Convergence criteria

They are sufficient condition to establish the convergence of a sequence

## Theorem (Comparison Criterion)

$\{x_n\}, \{y_n\}, \{z_n\}$  – real sequences  
 If eventually  $y_n \leq x_n \leq z_n$  and  $y_n \rightarrow L \in \mathbb{R}$  and  $z_n \rightarrow L \in \mathbb{R}$ , then  
 $x_n \rightarrow L \in \mathbb{R}$  or  $\lim_{n \rightarrow +\infty} x_n = L \in \mathbb{R}$

### Proof

Suppose  $L \in \mathbb{R}$ . Given  $\varepsilon > 0$ , two positions  $n_\varepsilon^1$  and  $n_\varepsilon^2$  exist such that  
 $y_n \in B_\varepsilon(L), \forall n \geq n_\varepsilon^1$  and  $z_n \in B_\varepsilon(L), \forall n \geq n_\varepsilon^2$   
 Suppose the existence of a position  $n^3$  such that  
 $y_n \leq x_n \leq z_n, \forall n \geq n^3$ . Taking  $n^* = \max\{n_\varepsilon^1, n_\varepsilon^2, n^3\}$  we have  

$$\begin{cases} L - \varepsilon < y_n < L + \varepsilon \\ L - \varepsilon < z_n < L + \varepsilon \Rightarrow L - \varepsilon < y_n \leq x_n \leq z_n < L + \varepsilon, \forall n \geq n^* \\ y_n \leq x_n \leq z_n \end{cases}$$
  
 $\Rightarrow L - \varepsilon < x_n < L + \varepsilon, \forall n \geq n^* \Rightarrow x_n \in B_\varepsilon(L), \forall n \geq n^*$   
 Therefore  $x_n \rightarrow L \in \mathbb{R}$

Eg.

$$\{x_n\}, \lim_{n \rightarrow +\infty} \frac{\sin n}{n^2}, \nexists \lim_{n \rightarrow +\infty} \sin n; \nexists \lim_{n \rightarrow +\infty} \cos n$$

NB.

$$\lim_{n \rightarrow +\infty} \frac{\sin n}{n^2} = \frac{\lim_{n \rightarrow +\infty} \sin n}{\lim_{n \rightarrow +\infty} n^2}$$

$$\forall n \in \mathbb{N}, |\sin n| \leq 1 \text{ or } -1 \leq \sin n \leq 1$$

$$\forall n \in \mathbb{N}, |\cos n| \leq 1 \text{ or } -1 \leq \cos n \leq 1$$

$$\Rightarrow -\frac{1}{n^2} \leq \frac{\sin n}{n^2} \leq \frac{1}{n^2} \Rightarrow \frac{\sin n}{n^2} \rightarrow 0$$

## Theorem (Ratio Criterion)

$\{x_n\}$  real sequence. If a scalar  $q < 1$  exists such that **eventually**

$$\left| \frac{x_{n+1}}{x_n} \right| < q \text{ then } \lim_{n \rightarrow +\infty} x_n = 0$$

NB

$$\text{If the ratio } \left| \frac{x_{n+1}}{x_n} \right| \text{ has a limit then } \lim_{n \rightarrow +\infty} \left| \frac{x_{n+1}}{x_n} \right| = q < 1$$

Then  $x_n$  is convergent to 0

$\rightarrow 0$   
1

Eg.

## Napier's constant

Show that  $\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n = e, \forall a > 1$  and  $k \in \mathbb{R}$

$$\frac{n^k}{a^n} > 0, \forall n \geq 0; x_{n+1} = \frac{(n+1)^k}{a^{n+1}}$$

Given  $\{x_n\}$  a ~~Cauchy~~ <sup>Cauchy</sup> sequence with  $x_n = \frac{(n+1)^k}{a^{n+1}}, n \geq 1$   
 $\frac{x_{n+1}}{x_n} = \frac{(n+2)^k}{a^{n+2}} \cdot \frac{a^{n+1}}{(n+1)^k} = \frac{1}{a} \left(\frac{n+2}{n+1}\right)^k = \frac{1}{a} \left(1 + \frac{1}{n+1}\right)^k = \frac{1}{a} = q < 1$

1.  $\{x_n\}$  is monotone increasing

2.  $\{x_n\}$  is bounded from above

$\Rightarrow \{x_n\}$  converges to  $L = \sup E$

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n = e \text{ or } \left(1 + \frac{1}{n}\right)^n = e$$

## Important limits

1. If  $|x_n| \rightarrow +\infty$  and  $k \in \mathbb{R}$

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{k}{x_n}\right)^{x_n} = e^k$$

Eg.

$$\begin{cases} \left(1 + \frac{2}{n^2}\right)^{n^2} \rightarrow e^2; \\ k = 2 \end{cases}; \begin{cases} \left(1 - \frac{1}{2\sqrt{n}}\right)^{\sqrt{n}} \rightarrow e^{-0.5} \\ k = -\frac{1}{2}, x_n = \sqrt{n} \end{cases}$$

2. If  $y_n \rightarrow 0$  and  $y_n > 0$  then

$$\lim_{n \rightarrow +\infty} (1 + y_n)^{\frac{1}{y_n}} = e$$

Eg.

$$y_n = 2^{-n^2} \rightarrow 0; \left(1 + 2^{-n^2}\right)^{\frac{1}{2^{-n^2}}} \rightarrow e$$

3. If  $y_n \rightarrow 0$  and  $y_n > 0$

$$\lim_{n \rightarrow +\infty} \frac{\ln(1 + y_n)}{y_n} = 1$$

NB

$$\lim_{n \rightarrow +\infty} \frac{\log_b(1 + y_n)}{y_n} = \log_b e, \forall b > 0, b \neq 0$$

$$\left\{ \begin{array}{l} \lim_{n \rightarrow +\infty} \frac{\ln\left(1 + \frac{1}{2n^2}\right)}{\frac{1}{2n^2}} = e \\ y_n = \frac{1}{2n^2} \rightarrow 0 \end{array} \right.$$

4. If  $c > 0, y_n \rightarrow 0$  and  $y > 0$  then

$$\lim_{n \rightarrow +\infty} \frac{c^{y_n} - 1}{y_n} = \ln c$$

$$[c = e, \lim_{n \rightarrow +\infty} \frac{e^{y_n} - 1}{y_n} = 1]$$

5. If  $a \in \mathbb{R}$  and  $y_n \rightarrow 0$  with  $y_n > 0$ , then

$$\lim_{n \rightarrow +\infty} \frac{(1 + y_n)^a - 1}{y_n} = a$$

Eg.

$$\begin{aligned} x_n &= \sqrt{n^2 + n} - n. \text{ Ind. form } +\infty - \infty \\ \sqrt{n^2 + n} - n &= \sqrt{n^2 \left(1 + \frac{1}{n}\right)} - n = n \left[ \left(1 + \frac{1}{n}\right)^{0.5} - 1 \right] \\ &= \frac{\left( \left(1 + \frac{1}{n}\right)^{0.5} - 1 \right)}{\frac{1}{n}} \rightarrow \frac{1}{2} \end{aligned}$$

## Orders of convergence and divergence

NB.

Some sequences converge to the same value with different speed.

Eg.

$$\left\{ \frac{1}{n} \right\} \wedge \left\{ \frac{1}{n^2} \right\} \rightarrow 0, \text{ but the latter diverges faster}$$

### Definition

$\{x_n\}, \{y_n\}$  real sequences with  $\{x_n\}$  eventually different from 0 ( $\exists n^* \geq 1: \forall n \geq n^*, x_n \neq 0$ )

1. If  $\frac{y_n}{x_n} \rightarrow 0$  then we say that  $\{y_n\}$  is negligible w.r.t.  $x_n: y_n = o(x_n)$

2. If  $\frac{y_n}{x_n} \rightarrow k$  then we say that  $\{y_n\}$  is comparable w.r.t.  $\{x_n\}: y_n \asymp x_n$

If  $k = 1$ , then,  $\frac{y_n}{x_n} = 1$  and  $\{x_n\}, \{y_n\}$  are said to be asymptotic sequences  $x_n \sim y_n$

### Lemma 352

$\{x_n\}, \{y_n\}$  real sequences with terms eventually different from zero

a. The relation of comparability  $\asymp$  (and  $\sim$ ) is reflexive, symmetric and transitive

b. The relation of negligibility ( $o$ ) is transitive

Proof

$$\text{a. } \frac{x_n}{x_n} \rightarrow 1; \frac{y_n}{x_n} \rightarrow k \neq 0 \Leftrightarrow \frac{x_n}{y_n} \rightarrow \frac{1}{k} \neq 0; \frac{z_n}{y_n} \rightarrow k \neq 0 \text{ and } \frac{y_n}{x_n} \rightarrow k \neq 0 \Rightarrow \frac{z_n}{x_n} = \frac{z_n}{y_n} * \frac{y_n}{x_n} \rightarrow k * \frac{1}{k} \neq 0$$

$$\text{b. } \frac{z_n}{y_n} \rightarrow 0, \frac{y_n}{x_n} \rightarrow 0 \Rightarrow \frac{z_n}{x_n} = \frac{z_n}{y_n} * \frac{y_n}{x_n} \rightarrow 0 \Rightarrow z_n = o(x_n)$$

### Proposition 353: Little-o algebra

$\{x_n\}, \{y_n\}$  real sequences and  $c \neq 0$

$$1. 0(x_n) + 0(x_n) = 0(x_n)$$

$$2. 0(x_n) + 0(y_n) = 0(x_n * y_n)$$

$$3. c * 0(x_n) = 0(x_n)$$

$$4. \text{ If } y_n = 0(x_n) \text{ then } 0(y_n) + 0(x_n) = 0(x_n)$$

Proof: If  $y_n = 0(x_n)$  then  $\{y_n\}$  can be written as follows

$x_n * \varepsilon_n$ , where  $\varepsilon_n$  is an infinitesimal sequence

that is  $y_n = \varepsilon_n * x_n$  with  $\varepsilon_n \rightarrow 0$

Eg.

$$\frac{1}{n^2} = 0\left(\frac{1}{n}\right)$$

$$\frac{1}{n^2} = \frac{1}{n} * \frac{1}{n}$$

1. Let's indicate the two little-0's with  $x_n * \varepsilon_n$  and  $x_n * \delta_n$ , we have  
 $0(x_n) + 0(x_n) = x_n \varepsilon_n + x_n \delta_n = (\varepsilon_n + \delta_n)x_n = \alpha_n x_n = 0(x_n)$

2. Let  $0(x_n) = \varepsilon_n x_n$  and  $0(y_n) = \delta_n y_n$   
 $0(x_n) * 0(y_n) = (\varepsilon_n x_n)(\delta_n y_n) = (\varepsilon_n \delta_n) * x_n y_n = \alpha_n x_n y_n = 0(x_n y_n)$

3. Let  $0(x_n) = \varepsilon_n x_n$   
 $c * 0(x_n) = c(\varepsilon_n x_n) = (c\varepsilon_n)x_n = 0(x_n)$

NB

$x_n \rightarrow 0$  then  $x_n = 0(1)$

$$\frac{1}{n} = 0(1) \Leftrightarrow \frac{1}{n} \rightarrow 0$$

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = \lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$

## Asymptotic equivalence

Given  $\{x_n\}, \{y_n\}$  real sequence. If  $x_n \sim y_n$  then  $\{x_n\}, \{y_n\}$  have the same limit  $L \in \bar{\mathbb{R}}$   
 $x_n \rightarrow L \in \bar{\mathbb{R}} \Leftrightarrow y_n \rightarrow L \in \bar{\mathbb{R}}$

### Lemma 355

Let  $y_n \sim x_n$  and  $z_n \sim w_n$ . Then

b.  $y_n * z_n \sim x_n * w_n$

c. Assume that  $z_n \neq 0$  and  $w_n \neq 0$ . Eventually

$$\frac{y_n}{z_n} \sim \frac{x_n}{w_n}$$

Proof

$$(b) \frac{y_n * z_n}{x_n * w_n} = \frac{y_n}{x_n} * \frac{z_n}{w_n} \rightarrow 1$$

$$(c) \frac{\frac{y_n}{z_n}}{\frac{x_n}{w_n}} = \frac{y_n}{z_n} * \frac{w_n}{x_n} = \frac{y_n}{x_n} * \frac{1}{\frac{z_n}{w_n}} \rightarrow 1$$

### Lemma 356

$\{x_n\}$  real sequence.  $x_n \sim x_n + 0(x_n)$

Proof

We have:

$$\frac{x_n + 0(x_n)}{x_n} = \frac{x_n}{x_n} + \frac{0(x_n)}{x_n} = 1 + \frac{0(x_n)}{x_n} \rightarrow 1$$

From this lemma 356, we have:

$$x_n + 0(x_n) \rightarrow L \Leftrightarrow x_n \rightarrow L$$

Therefore, we have:

$$1. (x_n + 0(x_n))(y_n + 0(y_n)) \sim x_n y_n$$

$$\text{Eg. } (n^2 + n + 10)(n^3 + \ln n - 1) = (n^2 + 0(n^2))(n^3 + 0(n^3)) = n^5$$

$$2. \frac{x_n + 0(x_n)}{y_n + 0(y_n)} \sim \frac{x_n}{y_n}$$

$$\text{Eg: } \frac{n^4 + 10n}{n^2 + \ln n} = \frac{n^4}{n^2} = n^2$$

### Proposition 359

$\{x_n\}, \{y_n\}$  real sequences

$x_n \sim y_n \Leftrightarrow x_n = y_n + 0(y_n)$

Proof:

( $\Leftarrow$ )

Assume that  $x_n = y_n + 0(y_n)$ , we have

$$\frac{x_n}{y_n} = \frac{y_n + 0(y_n)}{y_n} = \frac{y_n}{y_n} + \frac{0(y_n)}{y_n} = 1 + \frac{0(y_n)}{y_n} \rightarrow 1$$

Therefore,  $x_n \sim y_n$

( $\Rightarrow$ )

Assume that  $x_n \sim y_n$ . Taking  $z_n = x_n - y_n$ , we have

$$\frac{z_n}{y_n} = \frac{x_n - y_n}{y_n} = \frac{x_n}{y_n} - 1 \rightarrow 0$$

$$\Rightarrow z_n = 0(y_n)$$

$$\begin{aligned} \Rightarrow x_n - y_n &= 0(y_n) \\ \Rightarrow x_n &= y_n + 0(y_n) \end{aligned}$$

### Proposition 360

$\{x_n\}$  real sequence eventually non-zero

Then

$$\frac{1}{n} * \ln|x_n| \rightarrow k \neq 0 \Leftrightarrow |x_n| = e$$

Proof

( $\Leftarrow$ )

Assume that  $|x_n| = e^{kn+0(n)}$

$$\frac{1}{n} \ln|x_n| = \frac{1}{n} \ln e^{kn+0(n)} = \frac{1}{n} (kn + 0(n)) = k + \frac{0(n)}{n} \rightarrow k$$

( $\Rightarrow$ )

Assume that  $\frac{1}{n} \ln|x_n| \rightarrow k \neq 0$

$$|x_n| = e^{n \ln|x_n|} = e^{kn+0(n)}$$

$$\frac{1}{nk} \ln|x_n| \rightarrow 1 \Rightarrow \frac{\ln|x_n|}{nk} \rightarrow 1 \Rightarrow \ln|x_n| \sim nk \Rightarrow \ln|x_n| = nk + 0(nk)$$

### Terminology

- $\{x_n\}, \{y_n\}$  infinitesimals  $x_n \rightarrow 0, y_n \rightarrow 0$   
 $y_n = 0(x_n)$   
 $\{y_n\}$  is a higher of infinitesimal order w.r.t.  $\{x_n\}$   
 Eg.  $\frac{1}{n^3} = 0 * \frac{1}{n^2}$
- $\{x_n\}, \{y_n\}$  infinite sequences,  $x_n \rightarrow \pm\infty, y_n \rightarrow \pm\infty$   
 $y_n = 0(x_n)$   
 $\{y_n\}$  is a lower order of infinite w.r.t.  $\{x_n\}$   
 EG.  $n^2 = 0(n^3)$

### Scales of infinite and infinitesimal

- $\{a^n\}, a > 1$  (is decreasing order of base)
- $\{n^k\}, k > 0$  (is decreasing order of exponent)
- $\{(\ln n)^k\}, k > 0$  (in decreasing order of exponent)

$$n^k = 0(a^n), (\ln n)^k = 0(n^k), (\ln n)^k = 0(a^n)$$

Eg.

$$\frac{n^{10000}}{2^n} \rightarrow 0$$

- $\{a^n\}, 0 < a < 1$ , (in decreasing order of base)
- $\{n^k\}, k < 0$ , (in decreasing order of exponent)
- $\{(\ln n)^k\}, k < 0$  (in decreasing order of exponent)

$$a^n = 0(n^k); n^k = 0(\ln n)^k, a^n = 0[(\ln n)^k]$$

### Lemma 361

A.  $a^n = 0(n!), a > 1, \frac{a^n}{n!} \rightarrow 0$  or  $\frac{n!}{a^n} \rightarrow +\infty$

B.  $n! = 0(n^n), \frac{n!}{n^n} \rightarrow 0$

### Sequence in $R^n$

Let  $K$  be a positive integer. We can consider the following sequence:

$$\{x^k\} \text{ with } x^k = (x_1^k, x_2^k, \dots, x_n^k) \in R^n, k \geq 1$$

Which is called a **sequence of vectors** in  $R^n$

Eg.

$$x^k = \begin{bmatrix} 1 \\ \frac{1}{k} \\ 1 \\ k^2 \end{bmatrix}, k \geq 1, \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \\ 4 \end{pmatrix}, \dots \right\}$$

### Definition

We say that  $\{x^k\}$  tends to  $L \in R^n$ , i.e.  $x^k \rightarrow L \in R^n$  as  $k \rightarrow +\infty$ , if

$$\forall \varepsilon > 0, \exists k_\varepsilon \geq 1: k \geq k_\varepsilon \Rightarrow \|x^k - L\| < \varepsilon$$

Eg.

$$x^k = \begin{bmatrix} \frac{1}{k} \\ \frac{1}{2k} \end{bmatrix}, \lim_{k \rightarrow +\infty} x^k = o \in R^2, \begin{bmatrix} \frac{1}{k} \\ \frac{1}{2k} \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\forall \varepsilon > 0, \exists k_\varepsilon \geq 1: k \geq k_\varepsilon \Rightarrow \|x^k - 0\| < \varepsilon$$

1. We suppose the existence of  $k_\varepsilon$

$$\|x^k - 0\| = \sqrt{\left(\frac{1}{k} - 0\right)^2 + \left(\frac{1}{2k} - 0\right)^2} = \sqrt{\frac{1}{k^2} + \frac{1}{4k^2}} = \sqrt{\frac{5}{4k^2}}$$

$$\sqrt{\frac{5}{4k^2}} < \varepsilon \Rightarrow \frac{5}{4k^2} < \varepsilon^2 \Rightarrow \frac{1}{4k^2} < \frac{\varepsilon^2}{5} \Rightarrow 4k^2 > \frac{5}{\varepsilon^2} \Rightarrow k^2 > \frac{5}{4\varepsilon^2} \Rightarrow k > \sqrt{\frac{5}{4\varepsilon^2}} = \frac{\sqrt{5}}{2\varepsilon}$$

$$k_\varepsilon = \left\lceil \frac{\sqrt{5}}{2\varepsilon} \right\rceil + 1$$

2. Taking  $k_\varepsilon = \left\lceil \frac{\sqrt{5}}{2\varepsilon} \right\rceil + 1$  and assuming that  $k \geq k_\varepsilon$  ( $\frac{1}{k} \leq \frac{1}{k_\varepsilon}$ )

$$\|x^k - 0\| = \sqrt{\frac{1}{k^2} + \frac{1}{4k^2}} = \sqrt{\frac{5}{4k^2}} \leq \sqrt{\frac{5}{4k_\varepsilon^2}} = \frac{\sqrt{5}}{2k_\varepsilon} = \frac{\sqrt{5}}{2\left(\left\lceil \frac{\sqrt{5}}{2\varepsilon} \right\rceil + 1\right)^2}$$

$$< \frac{\sqrt{5}}{2\left(\left\lceil \frac{\sqrt{5}}{2\varepsilon} \right\rceil + 1 - 1\right)^2} = \frac{\sqrt{5}}{4 * \frac{\sqrt{5}}{4\varepsilon^2}} = \varepsilon^2 < \varepsilon$$

NB.

$$\|x^k - L\| \rightarrow 0 \Leftrightarrow |x_i^k - L_i| \rightarrow 0, \forall i$$

### Definition

Let  $\{x_n\}_{n \geq 1}$  be a real sequence. We build a new sequence from  $\{x_n\}$  as follows:

$$s_1 = x_1, s_2 = x_1 + x_2, s_3 = x_1 + x_2 + x_3, \dots, s_n = x_1 + x_2 + \dots + x_n$$

Which is called the **partial sum sequence**  $\{s_n\}$  associated with  $\{x_n\}$

Let  $\{x_n\}$  be a real sequence. The partial sum sequence  $\{s_n\}$  associated with  $\{x_n\}$  is called series

$$\sum_{n=1}^{+\infty} x_n - \text{Series with general term } x_n \text{ (can also be indicated as } \sum x_n)$$

Target: study the character of the series (convergent, divergent, oscillating)

Let  $\sum x_n$  be a series.  $\sum x_n$  is said to be:

1. Convergent with sum  $S$  ( $\sum x_n = S$ ) if  $s_n \rightarrow S \in \mathbb{R}$
2. Divergent to  $\pm\infty$  if  $s_n \rightarrow \pm\infty$
3. Irregular or oscillating if  $\nexists \lim_{n \rightarrow +\infty} s_n$

Eg.

$$\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} \text{ Using the definition (Mengoli's series)}$$

- $s_n = \sum_{k=1}^n \frac{1}{k(k+1)}$  - General term of the sequence of the partial sums associated with  $s_n$

- $\lim_{n \rightarrow +\infty} s_n$  - ?

NB.

$$1. \frac{1}{k(k+1)} = \frac{(k+1) - k}{k(k+1)} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$2. \frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$$

$$\frac{1}{k(k+1)} = \frac{(Ak + A + Bk)}{k(k+1)} \Rightarrow \frac{1}{k(k+1)} = \frac{(A+B) \cdot k + A}{k(k+1)}$$

$$\forall k \geq 1, 1 = (A+B) \cdot k + A \Rightarrow \begin{cases} A+B = 0 \\ A = 1 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -1 \end{cases}$$

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}, \forall k \geq 1$$

$$s_n = \sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n \left[ \frac{1}{k} - \frac{1}{k+1} \right]$$

$$= \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \dots + \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{n+1}$$

$$s_n = \sum_{k=1}^n \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} \xrightarrow{\rightarrow 0} 1 = S \in \mathbb{R}$$

Eg2

### Harmonic series:

$$\sum_{n=1}^{+\infty} \frac{1}{n}$$

We show that the Harmonic series is divergent to  $+\infty$

$$\bullet s_n = \sum_{k=1}^n \frac{1}{k}$$

NB

If  $\{x_n\}$  converges to a real number, then every subsequence of  $\{x_n\}$  converges to the same real number.

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}, \frac{1}{n} \rightarrow 0$$

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\right\}, \frac{1}{2n} \rightarrow 0$$

If we can find a subsequence such that it diverges to  $+\infty \Rightarrow \{x_n\}$  is divergent

Target:

we find a subsequence divergent to  $+\infty$  of  $s_n$

Subsequence with position  $n = 2^k, k \in \mathbb{N}, \begin{pmatrix} 1 & 2 & 4 & \dots \\ s_1 & s_2 & s_4 & \dots \end{pmatrix}$

$$s_1 = 1$$

$$s_2 = 1 + \frac{1}{2} \quad > \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + 2 * \frac{1}{2} = 2$$

$$s_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} > 1 + 2 * \frac{1}{2} + \frac{1}{2} = 1 + 3 * \frac{1}{2}$$

...

$$s_{2^k} > 1 + k * \frac{1}{2}$$

NB

$1 + k * \frac{1}{2}$  is unbounded

$\{s_{2^k}\}$  subsequence of  $\{s_n\}$  is monotone increasing and unbounded  $\Rightarrow \{s_{2^k}\}$  is divergent to  $+\infty \Rightarrow \{s_n\}$  is divergent to  $+\infty, \lim_{n \rightarrow +\infty} s_n = +\infty$

Therefore,  $s_n = \sum_{k=1}^{+\infty} \frac{1}{k} = +\infty$

NB.

The character of a series does not change if we don't consider a finite of terms

$$\sum_{k=1}^{+\infty} \frac{1}{k} = +\infty; \sum_{n=15}^{+\infty} \frac{1}{n} = +\infty; \sum_{n=4}^{+\infty} \frac{1}{n(n+1)} - \text{converges, but with a different sum}$$

$$\sum_{n=4}^{+\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{+\infty} \frac{1}{n(n+1)} - \sum_{n=1}^3 \frac{1}{n(n+1)} = 1 - \left[ \frac{1}{2} + \frac{1}{6} + \frac{1}{12} \right] = 1 - \frac{1}{2} - \frac{1}{6} - \frac{1}{12}$$

Eg.

## Geometric series

$$\sum_{n=1}^{+\infty} q^n, q \in \mathbb{R}$$

NB. The character of the series depends on q

$$s_n = \sum_{k=0}^n q^k$$

$$\bullet q = 1, s_n = \sum_{k=0}^n 1^k = \sum_{k=0}^n 1 = (n - 0 + 1) * 1 = n + 1 \rightarrow +\infty$$

therefore the geometric series is divergent to positive infinity

- $q \neq 1: S_n = \sum_{k=0}^n q^k = 1 + q + q^2 + \dots + q^n$ 

$$\begin{cases} S_n = 1 + q + q^2 + \dots + q^n \\ q * S_n = q + q^2 + \dots + q^{n+1} \\ S_n - qS_n = (1 + q + q^2 + \dots + q^n) - (q + q^2 + q^3 + \dots + q^{n+1}) \\ (1 - q)S_n = 1 - q^{n+1} \\ \lim_{n \rightarrow +\infty} \frac{1 - q^{n+1}}{1 - q} \end{cases}$$

NB

$$\{q^{n+1}\} = \begin{cases} q > 1 & q^{n+1} \rightarrow +\infty \\ -1 < q < 1 & q^{n+1} \rightarrow 0 \\ q \leq -1 & q^{n+1} \text{ irregular} \end{cases}$$

- $q > 1: \lim_{n \rightarrow +\infty} \frac{1 - q^{n+1}}{1 - q} = +\infty$

- $-1 < q < 1: \lim_{n \rightarrow +\infty} \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q}$

- $q \leq -1: \lim_{n \rightarrow +\infty} \frac{1 - q^{n+1}}{1 - q}, \nexists$

$$\sum_{n=1}^{+\infty} q^n = \begin{cases} +\infty, & \text{if } q \geq 1 \\ \frac{1}{1 - q}, & \text{if } -1 < q < 1 \\ \text{irregular if } q \leq -1 \end{cases}$$

Eg.

- $\sum_{n=1}^{+\infty} 3^n = +\infty$

- $\sum_{n=1}^{+\infty} \frac{1}{2} = \frac{1}{1 - \frac{1}{2}} = 2$

- $\sum_{n=1}^{+\infty} \left(-\frac{2}{3}\right)^n = \frac{1}{1 + \frac{2}{3}} = 0.6$

- $\sum_{n=1}^{+\infty} (-1)^n$  - Irregular sequence

- $\sum_{n=1}^{+\infty} \left(-\frac{7}{3}\right)^n$  - Irregular sequence

NB

Let  $q \in (-1, 1)$ . The geometric series  $\sum_{n=0}^{+\infty} q^n = \frac{1}{1 - q}$

$$\begin{cases} \sum_{m=p}^{+\infty} q^n = \frac{q^p}{1 - p} \\ p \in \mathbb{N} \end{cases}$$

$$\left\{ \begin{array}{l} \sum_{m=2}^{+\infty} \left(\frac{1}{2}\right)^m = \frac{\left(\frac{1}{2}\right)^2}{1 - \frac{1}{2}} = \frac{1}{2} \\ \text{Geo series } q = \frac{1}{2} \in (-1,1) - \text{Convergent series} \end{array} \right.$$

## Properties:

Let  $\sum x_n$  and  $\sum y_n$  be convergent series and  $c \in \mathbb{R}$

1.  $\sum c * x_n$  converges and  $\sum c * x_n = c * \sum x_n$
2.  $\sum (x_n + y_n)$  converges and  $\sum (x_n + y_n) = \sum x_n + \sum y_n$

Eg.

$$\sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = 2$$

$$\sum_{n=0}^{+\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1 - \frac{2}{3}} = 3$$

$$\sum_{n=0}^{+\infty} 3 * \left(\frac{1}{2}\right)^n = 3 * \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n = 3 * 2 = 6$$

$$\sum_{n=0}^{+\infty} \left[ \left(\frac{1}{2}\right)^n + \left(\frac{2}{3}\right)^n \right] = \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n + \sum_{n=0}^{+\infty} \left(\frac{2}{3}\right)^n = 2 + 3 = 5$$

Eg2

$$\sum_{n=0}^{+\infty} \frac{2^{n-1}}{3^{n+2}}$$

NB

$$x_n = \frac{2^{n-1}}{3^{n+2}} = \frac{2^n - 2^{-1}}{3^n * 3^2} = \left(\frac{2}{3}\right)^n * \frac{1}{2} * \frac{1}{9} = \frac{1}{18} \left(\frac{2}{3}\right)^n$$

$$\sum_{n=0}^{+\infty} \left(\frac{2}{3}\right)^n, q = \frac{2}{3} \in (-1,1) \rightarrow \text{Convergent } S = \frac{1}{1 - \frac{2}{3}} = 3$$

$$\sum_{n=0}^{+\infty} \frac{1}{18} \left(\frac{2}{3}\right)^n \text{ converges and } S = \frac{1}{18} * 3 = \frac{1}{6}$$

## Theorem 380

Let  $\sum x_n$  be a series

$$\sum x_n \text{ converges} \Rightarrow x_n \rightarrow 0$$

Proof

Assume that  $\sum x_n$  converges with sum  $S$ .

$$x_n = \sum_{k=1}^n x_k - \sum_{k=1}^{n-1} x_k \Rightarrow x_n = S_n - S_{n-1}$$

$$\lim_{n \rightarrow +\infty} x_n = \lim_{n \rightarrow +\infty} [S_n - S_{n-1}] = \lim_{n \rightarrow +\infty} S_n - \lim_{n \rightarrow +\infty} S_{n-1} = S - S = 0$$

Therefore,  $x_n \rightarrow 0$  as  $n \rightarrow +\infty$

NB

$x_n \rightarrow 0 \not\Rightarrow \sum x_n$  convergent

Example: Harmonic series

$x_n \not\rightarrow 0 \Rightarrow \sum x_n$  divergent

Eg

$$\sum_{n=1}^{+\infty} \ln\left(1 + \frac{1}{n}\right), x_n = \begin{cases} \ln\left(1 + \frac{1}{n}\right) \rightarrow 0 & \left\{ \begin{array}{l} \sum x_n \text{ converges} \\ \sum x_n \text{ diverges} \\ \sum x_n \text{ irregular} \end{array} \right. \\ \text{Zero-test (Necessary cond)} \end{cases}$$

$$S_n = \sum_{k=1}^n \ln\left(1 + \frac{1}{k}\right)$$

NB:  $\ln\left(1 + \frac{1}{k}\right) = \ln\left(\frac{k+1}{k}\right) = \ln(k+1) - \ln k$

$$S_n = \sum_{k=1}^n [\ln(k+1) - \ln k] = [\ln 2 - \ln 1] + [\ln 3 - \ln 2] + [\ln 4 - \ln 3] + \dots + [\ln(n+1) - \ln n]$$

$$= \ln 1 + \ln(n+1) \rightarrow +\infty$$

Also

$$\ln\left(1 + \frac{1}{n}\right) \sim \frac{1}{n}, \sum \frac{1}{n} = +\infty$$

Eg.

$$\sum_{n=1}^{+\infty} \frac{n^2 + n}{3n^2 + \sqrt{n}}, x_n = \frac{n^2 + n}{3n^2 + \sqrt{n}} \sim \frac{n^2}{3n^2} \rightarrow \frac{1}{3} \neq 0$$

From zero-test, the series doesn't converge

Eg.

$$\sum_{n=1}^{+\infty} \left[\frac{1}{\gamma} - 1\right]^n - \text{geometric series, } q = \frac{1}{\gamma} - 1, \gamma \in \mathbb{R} \neq 0$$

- $\frac{1}{\gamma} - 1 = 1 \Rightarrow \frac{1}{\gamma} = 2 \Rightarrow \gamma > 0$  and  $\gamma = \frac{1}{2}$

- $\frac{1}{\gamma} - 1 > 1 \Rightarrow \frac{1}{\gamma} > 2 \left\{ \begin{array}{l} \gamma < 0 \text{ Impossible} \\ \gamma > 0, \gamma < \frac{1}{2} \end{array} \right.$

- $\left|\frac{1}{\gamma} - 1\right| < 1 \Rightarrow -1 < \frac{1}{\gamma} - 1 < 1 \Rightarrow 0 < \frac{1}{\gamma} < 2$

$$\left\{ \begin{array}{l} \gamma > 0 \\ \frac{1}{\gamma} < 2 \end{array} \right. \text{ or } \left\{ \begin{array}{l} \gamma < 0 \\ \text{Impossible} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \gamma > 0 \\ \gamma > \frac{1}{2} \end{array} \right. - \text{Series is convergent}$$

- $\frac{1}{\gamma} - 1 \leq -1 \Rightarrow \frac{1}{\gamma} \leq 0 \Rightarrow \gamma < 0$  - Irregular series

NB.

If  $\gamma > \frac{1}{2}$  then the series converges and  $S = \frac{\left(\frac{1}{\gamma} - 1\right)^1}{1 - \left(\frac{1}{\gamma} - 1\right)}$

## Series with positive terms

Let  $\sum x_n$  be a series. If  $x_n \geq 0, \forall n$  or eventually then the series is said to be a series with

Eg.

$$\sum_{n=1}^{+\infty} \frac{2^n}{n!}, \forall n \geq 1, \frac{2^n}{n!} > 0$$

$$\sum_{n=1}^{+\infty} \frac{n^2 - 100}{2n^2 + 10n}, \forall n \geq 10, \text{ series with eventually positive terms}$$

We can write the sequence of  $\{S_n\}$  using a recurrence formula:

$$\begin{cases} x_1 \in R \\ S_n = S_{n-1} + x_n, \forall n \geq 2 \end{cases}$$

NB.

$$S_n - S_{n-1} = x_n \geq 0 \Rightarrow S_n - S_{n-1} \geq 0 \Rightarrow S_n \geq S_{n-1}$$

$\{S_n\}$  is monotone increasing

1. If  $\{S_n\}$  is bounded (from above) then  $\sum x_n$  converges
2. If  $\{S_n\}$  is unbounded (from above) then  $\sum x_n$  diverges to  $+\infty$

### Proposition 382

A  $\sum x_n$  with positive terms is either convergent or divergent to  $+\infty$ .  
 $\sum x_n$  converges  $\Leftrightarrow \{S_n\}$  is bounded from above

### Proposition 383 (Comparison criterion)

Consider  $\sum x_n, \sum y_n$  with positive terms and  $x_n \leq y_n$  eventually.

1.  $\sum x_n = +\infty \Rightarrow \sum y_n = +\infty$
2.  $\sum y_n$  converges  $\Rightarrow \sum x_n$  converges

Eg.

$$\sum_{n=1}^{+\infty} \frac{2^n + n}{3^n + n^2}, x_n = \frac{2^n + n}{3^n + n^2} \geq 0, \forall n \geq 1$$

$$x_n = \frac{2^n + n}{3^n + n^2} \leq \frac{2^n + 2^n}{3^n} = \frac{2 * 2^n}{3^n} = 2 \left(\frac{2}{3}\right)^n, \sum_{n=1}^{+\infty} 2 \left(\frac{2}{3}\right)^n, q = \frac{2}{3} \in (-1, 1) \Rightarrow \text{series converges}$$

By the comparison criterion, the starting series converges.

Eg.

$$\sum_{n=1}^{+\infty} \frac{4n + 1}{3n^2 + n}, x_n = \frac{4n + 1}{3n^2 + n} \geq 0, x_n \rightarrow 0$$

$$x_n = \frac{4n + 1}{3n^2 + n} \geq \frac{4n}{3n^2 + n} \geq \frac{4n}{4n^2} = \frac{1}{n} = y_n, \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$$

By comparison criterion,  $\sum x_n = +\infty$ 

Eg.

$$\sum_{n=0}^{+\infty} \frac{\ln n}{n}, x_n = \frac{\ln n}{n} \geq 0, x_n \rightarrow 0$$

$$\left\{ \begin{array}{l} x_n = \frac{\ln n}{n} \geq \frac{1}{n} (\forall n \geq 3), \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty \Rightarrow \text{By comparison criterion, } \sum x_n \text{ diverges to } +\infty \\ \ln n \geq 1, \forall n \geq 3 \end{array} \right.$$

Eg.

$$\sum_{n=1}^{+\infty} \frac{1}{n!}, x_n = \frac{1}{n!} \geq 0, x_n \rightarrow 0, \text{Zero test}$$

NB.

$$1. \sum_{n=0}^{+\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \sum_{n=2}^{+\infty} \frac{1}{n!} = 2 + \sum_{n=2}^{+\infty} \frac{1}{n!} = 2 + \sum_{n=1}^{+\infty} \frac{1}{(n+1)!}$$

$$2. \frac{1}{(n+1)!} = \frac{1}{(n+1) * n!} = \frac{1}{(n+1) * n * (n-1) * \dots * 3 * 2 * 1} \leq \frac{1}{n(n+1)}$$

We know that  $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$  converges (with sum 1). By comparison criterion,

$$\sum_{n=1}^{+\infty} \frac{1}{(n+1)!}, \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} \frac{1}{(n+1)!} + 2 \text{ converges} \Rightarrow \sum_{n=0}^{+\infty} \frac{1}{n!} \text{ converges}$$

## Generalized Harmonic Series

$$\sum_{n=1}^{+\infty} \frac{1}{n^a} = \begin{cases} \text{Diverges to } +\infty \text{ if } a \leq 1 \\ \text{Converges with sum } S > 0 \text{ if } a > 1 \end{cases}$$

Eg.

$$- \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{+\infty} \frac{1}{n^{\frac{1}{2}}} = +\infty;$$

$$- \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{+\infty} \frac{1}{n^{\frac{3}{2}}} = S > 0$$

$$- a = 1: \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$$

$$- a < 1: \text{we know that } n^a < n^1 \Rightarrow \frac{1}{n^a} > \frac{1}{n}, \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$$

By comparison criterion, the  $\sum_{n=1}^{+\infty} \frac{1}{n^a} = +\infty$

Eg.

$$a = 2, \sum_{n=1}^{+\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{+\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{+\infty} \frac{1}{(n+1)^2}$$

NB

$$\left\{ \begin{array}{l} \frac{1}{(n+1)^2} = \frac{1}{(n+1)(n+1)} \leq \frac{1}{n(n+1)}, \sum_{n=1}^{+\infty} \frac{1}{n(n+1)} \text{ converges with sum } 1 \\ \frac{1}{n+1} \leq \frac{1}{n} \end{array} \right.$$

By comparison criterion,  $\sum_{n=1}^{+\infty} \frac{1}{(n+1)^2}$  converges  $\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{(n+1)^2} + 1$  converges  $\Rightarrow$

$$\sum_{n=2}^{+\infty} \frac{1}{n^2} \text{ converges}$$

Eg

$a > 2$ : we know that  $n^2 < n^a \Rightarrow \frac{1}{n^2} > \frac{1}{n^a}, \forall n \geq 1$

But  $\sum_{n=1}^{+\infty} \frac{1}{n^2}$  converges and, using the comparison criterion  $\sum_{n=1}^{+\infty} \frac{1}{n^a}$  converges

Eg.

$$\sum_{n=1}^{+\infty} \frac{n^2 + 1}{n^4 + \ln n + 10}, x_n \geq 0, x_n \sim \frac{n^2}{n^4} = \frac{1}{n^2} \rightarrow 0$$

$$x_n = \frac{n^2 + 1}{n^4 + \ln n + 10} \leq \frac{2n^2}{n^4} = \frac{2}{n^2} = 2 \frac{1}{n^2}, \sum_{n=1}^{+\infty} 2 \frac{1}{n^2} \text{ converges}$$

From the comparison criterion,  $\sum x_n$  converges

## Proposition 389 (Asymptotic Comparison Criterion)

Let  $\sum x_n$  and  $\sum y_n$  be series with strictly positive terms  
If  $x_n \sim y_n$  then the two series have the same character

Eg

$$\sum_{n=1}^{+\infty} \frac{2^n + 2n}{3^n + n^2}, x_n = \frac{2^n + 2n}{3^n + n^2} \sim \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n, \sum_{n=1}^{+\infty} \left(\frac{2}{3}\right)^n \quad q = \frac{2}{3} \in (-1, 1)$$

$$\sum_{n=1}^{+\infty} \left(\frac{2}{3}\right)^n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} \frac{2^n + 2n}{3^n + n^2}$$

Eg.

$$\sum_{n=1}^{+\infty} \frac{4n + 1}{3n^2 + n}, x_n = \frac{4n + 1}{3n^2 + n} \sim \frac{4n}{3n^2} = \frac{4}{3} * \frac{1}{n}, \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$$

NB

While studying the character of series, we have several ways:

$$\left\{ \begin{array}{l} 1. \text{ Definition } \lim_{n \rightarrow +\infty} S_n \\ 2. \text{ Convergence criteria } \left\{ \begin{array}{l} \text{comarison criterion} \\ \text{asymptotic comp. criterion} \\ \text{Ratio criterion} \end{array} \right. \end{array} \right.$$

## Proposition 393 (Ration criterion)

Let  $\sum x_n$  be a series with strictly positive terms eventually

Suppose that  $\lim_{n \rightarrow +\infty} \frac{x_{n+1}}{x_n}$  exists. We have

1. If  $\lim_{n \rightarrow +\infty} \frac{x_{n+1}}{x_n} < 1$  then  $\sum x_n$  converges

2. If  $\lim_{n \rightarrow +\infty} \frac{x_{n+1}}{x_n} > 1$  then  $\sum x_n$  diverges

3. If  $\lim_{n \rightarrow +\infty} \frac{x_{n+1}}{x_n}$

= 1 then it is impossible to snawer about the character of the series

$\Rightarrow$  We have to use another convergence criterion

Eg.

$$\sum_{n=1}^{+\infty} \frac{2^n}{n!}, x_n > 0, \forall n \geq 1, x_n \rightarrow +\infty, 2^n = O(n!)$$

1. Comparison criterion:  $x_n = \frac{2^n}{n!} \leq \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$ ,  $\sum_{n=1}^{+\infty} \left(\frac{2}{3}\right)^n$  converges
2. Ratio criterion:

$$\frac{x_{n+1}}{x_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{n!}{(n+1)!} * \frac{2^{n+1}}{2^n} = \frac{2}{n+1} \rightarrow 0 < 1, \text{ hence the series converges}$$

NB.

$$\sum_{n=1}^{+\infty} \frac{1}{n} = +\infty, \lim_{n \rightarrow +\infty} \frac{x_{n+1}}{x_n} = 1; \sum_{n=1}^{+\infty} \frac{1}{n^2}, \lim_{n \rightarrow +\infty} \frac{x_{n+1}}{x_n} = 1,$$

## Proposition 394

1. If a number  $q < 1$  exists such that  $\frac{x_{n+1}}{x_n} \leq q$  eventually then the series converges
2. If a number  $q > 1$  exists such that  $\frac{x_{n+1}}{x_n} \geq q$  eventually then the series converges

### Proof

1. Suppose  $\forall x \geq 1, x_n > 0$  and  $q < 1$  exists s.t.  $x_{n+1} \leq q * x_n$  we have

$$x_2 \leq q * x_1, x_3 \leq q * x_2 \leq q^2 x_1 \dots x_n \leq q^{n-1} x_1$$

Therefore  $0 < x_n \leq q^{n-1} x_1$ ,  $\sum q^{n-1} x_1$  converges  $\Rightarrow$  by comparison criterion the  $\sum x_n$  converges

2. Suppose  $\forall n \geq 1, x_n > 0$  and  $q > 1$  exists such that  $x_{n+1} \geq q x_n$

$$x_2 \geq q * x_1, x_3 \geq q * x_2 \geq q^2 x_1 \dots x_n \geq q^{n-1} x_1$$

Therefore  $0 < q^{n-1} x_1 \leq x_n$  and  $\sum q^{n-1} x_1 = +\infty \Rightarrow$  by comparison criterion the  $\sum x_n = +\infty$

Eg.

Discuss the character of

$$\sum_{n=0}^{+\infty} \frac{x^n}{n!}, x > 0$$

$$\left\{ \begin{array}{l} x_n = \frac{x^n}{n!} \rightarrow \text{Zero-test holds} \\ x^n = O(n!) \end{array} \right. \text{ We need to use a convergence criterion : ratio criterion}$$

$$\frac{x_{n+1}}{x_n} = \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{n!}{(n+1)!} * \frac{x^{n+1}}{x^n} = \frac{x}{n+1} \rightarrow_{n \rightarrow +\infty} 0$$

$$L = 0 < 1 \Rightarrow \sum_{n=0}^{+\infty} \frac{x^n}{n!} \text{ converges}$$

### Theorem 398

$$\sum_{n=0}^{+\infty} \frac{1}{n!} = e$$

NB.

$$\sum_{n=0}^{+\infty} \frac{1}{n!}$$

$$1. \sum_{n=0}^{+\infty} \frac{1}{n!} \text{ converges}$$

$$2. S = e$$

Eg.

$$\sum_{n=1}^{+\infty} \frac{1}{n!} \text{ converges, } \sum_{n=1}^{+\infty} \frac{1}{n!} + \frac{1}{0!} - \frac{1}{0!} = \sum_{n=0}^{+\infty} \frac{1}{n!} - 1 = e - 1$$

Eg.

$$\sum_{n=2}^{+\infty} \frac{2}{n!} = 2 \sum_{n=2}^{+\infty} \frac{1}{n!} = 2 \left[ \sum_{n=0}^{+\infty} \frac{1}{n!} + \frac{1}{0!} + \frac{1}{1!} - \frac{1}{0!} - \frac{1}{1!} \right] = 2 \left[ \sum_{n=0}^{+\infty} \frac{1}{n!} - 1 - 1 \right] = 2(e - 2)$$

Theorem 399

$\forall x \in R,$  we have

$$e^x = \lim_{n \rightarrow +\infty} \left[ 1 + \frac{x}{n} \right]^n = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

Eg.

$$\sum_{n=0}^{+\infty} \frac{2^n}{n!} = e^2;$$

$$\sum_{n=0}^{+\infty} \frac{2^{-n}}{n!} = \sum_{n=0}^{+\infty} \frac{1^n}{n!} = e^{\frac{1}{2}} = \sqrt{e}$$

$$\sum_{n=0}^{+\infty} \frac{-1^n}{n!} = e^{-1} = \frac{1}{e}$$

NB

We can observe that we have the series expansion of the exponential function

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n!} + \dots$$

Theorem 400: the Napier's constant  $e$  is an irrational number  
Series with terms of any sign

Eg.

$$\sum_{n=0}^{+\infty} \frac{n^2 - 100}{n^3 + 2n + 10}, x_n \text{ does not keep the same positive sign } \forall n \geq 1$$

$x_n$  eventually keeps the same positive sign ( $\forall n \geq 10$ )

$$\sum_{n=10}^{+\infty} \frac{n^2 - 100}{n^3 + 2n + 10}, x_n \geq 0, \forall n \geq 10; \frac{n^2 - 100}{n^3 + 2n + 10} \rightarrow 0$$

$$x_n = \frac{n^2 - 100}{n^3 + 2n + 10} \sim \frac{n^2}{n^3} = \frac{1}{n} = y_n, \sum_{n=0}^{+\infty} \frac{1}{n} = +\infty$$

$$\sum_{n=0}^{+\infty} x_n \text{ diverges to } +\infty \Rightarrow \sum_{n=10}^{+\infty} \frac{n^2 - 100}{n^3 + 2n + 10} = +\infty$$

- $x_n = \frac{n^2 - 100}{n^3 + 2n + 10} \dots$  (at home)

NB.

$$\sum_{n=1}^{+\infty} \frac{\sin n}{n^3 + n}; \sum_{n=1}^{+\infty} \frac{\cos(2n)}{n^3 + \ln n}; \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^3 + 2n} \leftarrow \text{Those series don't keep the same sign}$$

$$|x_n| = \left| \frac{\sin n}{n^3 + n} \right| \geq 0 - \text{Series with positive terms, we can use convergence criterion}$$

Def:

$\sum x_n$  series. The series  $\sum x_n$  is said to be **absolutely convergent** if the  $\sum |x_n|$  is convergent.

$$\sum_{n=0}^{+\infty} |x_n| \text{ convergent} \Rightarrow \sum_{n=1}^{+\infty} x_n \text{ is absolutely convergent}$$

Eg.

$$\sum_{n=1}^{+\infty} \frac{\sin n}{n^3 + n}, x_n = \frac{\sin n}{n^3 + n} \rightarrow 0, x_n = \frac{1}{n^3 + n} * \frac{\sin n}{1} \rightarrow 0$$

*→ 0*  
*bounded*

$$0 \leq |x_n| = \left| \frac{\sin n}{n^3 + n} \right| = \frac{|\sin n|}{|n^3 + n|} = \frac{|\sin n|}{n^3 + n} \leq \frac{1}{n^3 + n} \sim \frac{1}{n^3} = y_n$$

$$\sum_{n=1}^{+\infty} \frac{1}{n^3} \text{ convergent}; \sum_{n=1}^{+\infty} |x_n| \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} x_n \text{ is absolutely convergent}$$

Theorem 402

Let  $\sum x_n$  be a series

If  $\sum x_n$  is absolutely convergent, then  $\sum x$  is convergent.

Eg.

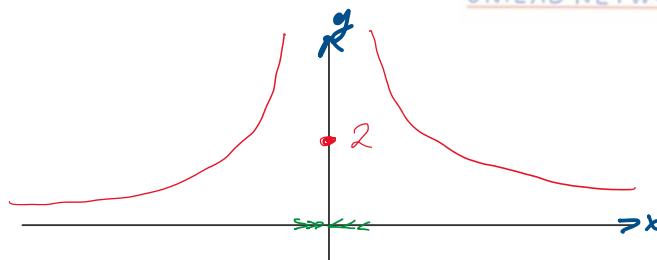
$$\sum_{n=1}^{+\infty} \frac{\sin n}{n^3 + n} \text{ is absolutely convergent} \Rightarrow \sum_{n=1}^{+\infty} \frac{\sin n}{n^3 + n} \text{ is convergent}$$

# Chapter 12 - Limits of function

giovedì 12 ottobre 2023 14:23

Eg.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

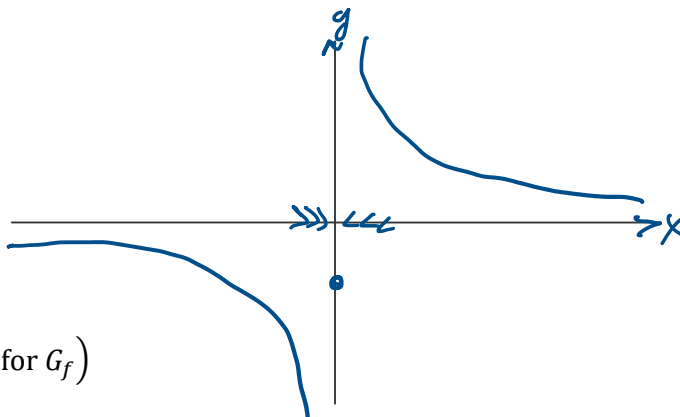


$$\lim_{x \rightarrow \pm\infty} f(x) = 0^+ \quad (y = 0 \text{ Horizontal asymptot of } G_f)$$

$$\begin{cases} \lim_{x \rightarrow 0^+} f(x) = +\infty \\ \lim_{x \rightarrow 0^-} f(x) = +\infty \end{cases}, \lim_{x \rightarrow 0} f(x) = +\infty$$

Eg.

$$g: \mathbb{R} \rightarrow \mathbb{R}, g(x) = \begin{cases} \frac{1}{x^3} & \text{if } x \neq 0 \\ -2 & \text{if } x = 0 \end{cases}$$



$$\begin{cases} \lim_{x \rightarrow +\infty} f(x) = 0^+ \\ \lim_{x \rightarrow -\infty} f(x) = 0^- \end{cases} - (y = 0 \text{ horiz. as. for } G_f)$$

$$\begin{cases} \lim_{x \rightarrow 0^-} f(x) = -\infty \\ \lim_{x \rightarrow 0^+} f(x) = +\infty \end{cases} \rightarrow (\text{"there is no limit of } f(x) \text{ for } x \rightarrow 0")$$

## We consider four different cases of calculation of limits for $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ :

1.  $\lim_{x \rightarrow x_0} f(x) = L, x_0 \in \mathbb{R}, L \in \mathbb{R}$
2.  $\lim_{x \rightarrow x_0} f(x) = \pm\infty$
3.  $\lim_{x \rightarrow \pm\infty} f(x) = L \in \mathbb{R}$
4.  $\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty$

### Definition

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$  and  $x_0 \in \mathbb{R}$  an accumulation point of  $A$ . We write

$$\lim_{x \rightarrow x_0} f(x) = L \in \mathbb{R} \text{ or } f(x) \rightarrow L \in \mathbb{R} \text{ as } x \rightarrow x_0$$

if

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0: \forall x \in A, 0 < |x - x_0| < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon$$

and  $L$  is called the limit of the function  $f$  as  $x \rightarrow x_0$

NB

$$0 < |x - x_0| < \delta_\varepsilon \text{ or } |x - x_0| < \delta_\varepsilon \text{ and } x \neq x_0$$

Eg.

$$\lim_{x \rightarrow 2} (2x + 5) = 9$$

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0: \forall x \in \mathbb{R}, 0 < |x - 2| < \delta_\varepsilon \Rightarrow |(2x + 5) - 9| < \varepsilon$$

1. Suppose the existence of  $\delta_\varepsilon$

$$|(2x + 5) - 9| = |2x - 4| = |2(x - 2)| = 2|x - 2| < 2\delta_\varepsilon = \varepsilon$$

$$\delta_\varepsilon = \frac{\varepsilon}{2} > 0$$

$$2\delta_\varepsilon \leq \varepsilon \Rightarrow \delta_\varepsilon \leq \frac{\varepsilon}{2} \Rightarrow 0 < \delta_\varepsilon \leq \frac{\varepsilon}{2}$$

2. Taking  $\delta_\varepsilon = \frac{\varepsilon}{2}$  and assuming that  $0 < |x - 2| < \delta_\varepsilon$

$$|(2x + 5) - 9| = |2x - 4| = 2|x - 2| < 2\delta_\varepsilon = 2 * \frac{\varepsilon}{2} = \varepsilon$$

Therefore  $|(2x + 5) - 9| < \varepsilon$

Taking  $\delta_\varepsilon = \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2}$  and assuming that  $0 < |x - 2| < \delta_\varepsilon$

$$|(2x + 5) - 9| = |2x - 4| = 2|x - 2| < 2\delta_\varepsilon = 2 * \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

Therefore  $|(2x + 5) - 9| < \varepsilon$

NB

Definition through radius:

$$V_{\delta_\varepsilon}(x_0) \text{ N-hood of } V_{\delta_\varepsilon}(x_0) = (x_0 - \delta_\varepsilon, x_0 + \delta_\varepsilon)$$

$x_0$  and radius be  $\delta_\varepsilon$

$$V_\varepsilon(L) \text{ N-hood of } L \text{ and radius } \varepsilon.$$

Def.

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$  and  $x_0 \in \mathbb{R}$  is an accumulation point of  $A$ . We write

$$\lim_{x \rightarrow x_0} f(x) = L \in \mathbb{R} \text{ or } f(x) \rightarrow L \in \mathbb{R} \text{ as } x \rightarrow x_0$$

if

$$\forall V_\varepsilon(L), \exists U_{\delta_\varepsilon}(x_0): \forall x \in U_{\delta_\varepsilon}(x_0) \cap A \text{ and } x \neq x_0 \Rightarrow f(x) \in V_\varepsilon(L)$$

And  $L$  is called the limit of  $f$  as  $x \rightarrow x_0$ .

Eg.

$$\lim_{x \rightarrow 2} (2x + 5) = 9$$

$$\forall V_\varepsilon(9), \exists U_{\delta_\varepsilon}(2): \forall x \in U_{\delta_\varepsilon}(2) \text{ and } x \neq 2 \Rightarrow 2x + 5 \in V_\varepsilon(9)$$

We can generalize with a unique definition for all the four cases of limits.

Definition

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$  and  $x_0 \in \bar{\mathbb{R}}$  is an accumulation point of  $A$ . We write

$$\lim_{x \rightarrow x_0} f(x) = L \in \bar{\mathbb{R}} \text{ or } f(x) \rightarrow L \in \bar{\mathbb{R}} \text{ as } x \rightarrow x_0$$

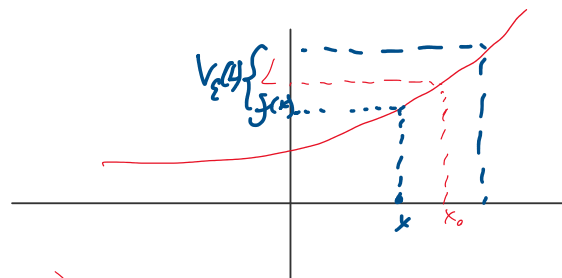
if

$$\forall V_\varepsilon(L), \exists U_{\delta_\varepsilon}(x_0): \forall x \in U_{\delta_\varepsilon}(x_0) \cap A \text{ and } x \neq x_0 \Rightarrow f(x) \in V_\varepsilon(L)$$

and  $L$  is called the limit of  $f$  as  $x \rightarrow x_0$

Eg.

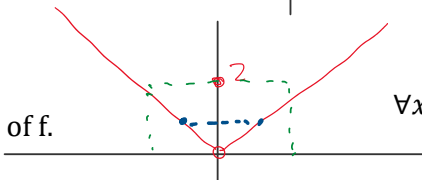
$$\lim_{x \rightarrow x_0} f(x) = L; x_0, L \in \mathbb{R}$$



Eg.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

$f$  is not convex,  $x_0 = 0$  is a local maximizer of  $f$ .



$$\lim_{x \rightarrow 0^-} f(x) = 0^+, \lim_{x \rightarrow 0^+} f(x) = 0^+ \Rightarrow \lim_{x \rightarrow 0} f(x) = 0$$

$$\text{NB } \left\{ \begin{array}{l} \lim_{x \rightarrow x_0} f(x) = L \in \mathbb{R} \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x), \lim_{x \rightarrow x_0^-} f(x) \text{ exist with some value} \\ x_0 \in \mathbb{R} \end{array} \right.$$

Now we consider  $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$

Eg.  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x_1, x_2) = \begin{cases} \frac{x_1}{x_1^2 + x_2^2} (x_1, x_2) \neq 0 \\ -1 \text{ if } (x_1, x_2) = 0 \end{cases}$$

$$\lim_{(x_1, x_2) \rightarrow (0,0)} f(x) \text{ or } \lim_{(x_1, x_2) \rightarrow (0,0)} \frac{x_1}{x_1^2 + x_2^2}$$

Def.

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  and  $x_0 \in \mathbb{R}^n$  is an accumulation point of  $A$ . We write

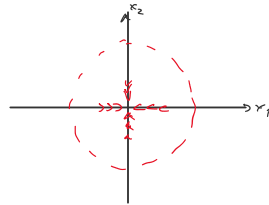
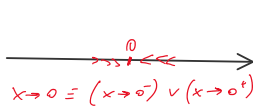
$$\lim_{x \rightarrow x_0} f(x) = L \in \bar{\mathbb{R}} \text{ or } f(x) \rightarrow L \in \bar{\mathbb{R}} \text{ as } x \rightarrow x_0$$

if

$V_\varepsilon(L), \exists U_{\delta_\varepsilon}(x_0): \forall x \in U_{\delta_\varepsilon}(x_0) \cap A \text{ and } x \neq x_0 \Rightarrow f(x) \in V_\varepsilon(L)$   
and  $L$  is called the limit of  $f$  as  $x \rightarrow x_0$

NB

$$\lim_{x \rightarrow 0} (x^2 + 1) \text{ and } \lim_{(x_1, x_2) \rightarrow (0,0)} \frac{x_1^2}{x_1^2 + x_2^2}$$



$$\lim_{x \rightarrow 0} (x^2 + 1) = 1$$

For any trajectory passing through the point  $x^0 = 0$  the value of the limit must be always the same.

If we find two different trajectories with two different limits then the limit of  $f(x)$  as  $x \rightarrow x_0$  does not exist

**Trajectory:**

$$x^1 = \left\{ \left( \frac{1}{n}, 0 \right) \right\}_{n \geq 1}, x^1 \rightarrow 0, f(x^1) = f\left(\frac{1}{n}, 0\right) = \frac{\left(\frac{1}{n}\right)^2}{\left(\frac{1}{n}\right)^2 + 0^2} = 1 \rightarrow 1$$

$$x^2 = \left\{ \left( \frac{1}{n}, \frac{1}{n} \right) \right\}_{n \geq 1}, x^2 \rightarrow 0, f(x^2) = f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{\left(\frac{1}{n}\right)^2}{\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^2} = \frac{1}{2} \rightarrow \frac{1}{2}$$

The limit doesn't exist (strategy 1: using sequences of vectors of  $\mathbb{R}^2$ )

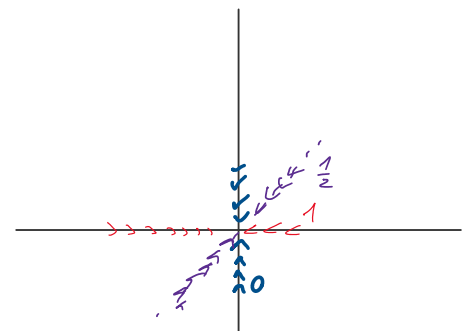
**Strategy 2**

$\lim_{x \rightarrow 0} \frac{x_1^2}{x_1^2 + x_2^2}$  does not exist

$$x_2 = 0 (x_1 \text{ Axis}): f(x_1, 0) = \frac{x_1^2}{x_1^2 + x_2^2} = \frac{x_1^2}{x_1^2} = 1 \rightarrow 1$$

$$x_1 = 0 (x_2 \text{ Axis}): f(0, x_2) = \frac{x_1^2}{x_1^2 + x_2^2} = 0 \rightarrow 0$$

$$x_1 = x_2 (\text{Bisector line}): f(x_1, x_2) = \frac{x_1^2}{x_1^2 + x_2^2} = \frac{1}{2} \rightarrow \frac{1}{2}$$



$$x_1 = 0 \text{ (} x_2 \text{ Axis): } f(0, x_2) = \frac{0}{x_1^2 + x_2^2} = 0 \rightarrow 0$$

$$x_1 = x_2 \text{ (Bisector line): } f(x_1, x_2) = \frac{x_1^2}{x_1^2 + x_2^2} = \frac{1}{2} \rightarrow \frac{1}{2}$$

### Proposition 528

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  and  $x_0 \in \mathbb{R}^n$  is an accumulation point of  $A$ . We have

$$\lim_{x \rightarrow x_0} f(x) = L \in \bar{\mathbb{R}} \Leftrightarrow x_n \rightarrow x_0 \Rightarrow f(x_n) \rightarrow L$$

For any sequence  $\{x_n\}$  of  $A$  with terms different from  $x_0$

Eg.

$$\lim_{x \rightarrow 2} (3x - 5) = 1$$

Let  $\{x_n\}$  be any sequence of  $\mathbb{R}$  with  $x_n \neq 2 \forall n \geq 1$  such that  $x_n \rightarrow 2$

$$f(x_n) = 3x_n - 5 \rightarrow 3 * 2 - 5 = 1 \Rightarrow \lim_{x \rightarrow 2} (3x - 5) = 1$$

NB. If we can find two different sequences of points of  $A$ , i.e.  $\{x_n\}$  and  $\{y_n\}$  such that

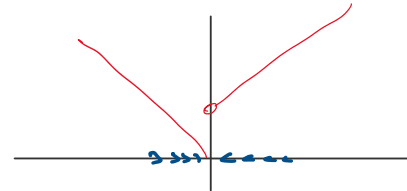
$x_n \neq x_0$  and  $y_n \neq x_0 \forall n \geq 1, x_n \rightarrow x_0$  and  $y_n \rightarrow y_0$

$$f(x_n) \rightarrow L_1 \text{ and } f(y_n) \rightarrow L_2$$

and  $L_1 \neq L_2$ , therefore the  $\lim_{x \rightarrow x_0} f(x)$  does not exist

Eg.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} -x, & x \leq 0 \\ x + 1, & x > 0 \end{cases}$$



Show that  $\nexists \lim_{x \rightarrow 0} f(x)$

$$x_n = \frac{1}{n} \rightarrow 0^+, y_n = -\frac{1}{n^2} \rightarrow 0^-$$

$$f(x) = \frac{1}{n} + 1 \rightarrow 1$$

$$f(y) = \left(-\frac{1}{n^2}\right) = \frac{1}{n^2} \rightarrow 0$$

Since  $L_1 \neq L_2 \Rightarrow \nexists \lim_{x \rightarrow 0} f(x)$

### Theorem 531 (Uniqueness of the limit)

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  and  $x_0 \in \mathbb{R}^n$  an accumulation point of  $A$ . There exists at most a unique  $L \in \bar{\mathbb{R}}$  such that

$$\lim_{x \rightarrow x_0} f(x) = L$$

### Theorem 532 (permanence of sign)

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  and  $x_0 \in \mathbb{R}^n$  an accumulation point of  $A$

If  $\lim_{x \rightarrow x_0} f(x) = L \neq 0$  then a  $B_\varepsilon(x_0)$  exists such that  $f(x)$  and  $L$  have the same sign, i.e.

$$f(x) * L > 0, \forall x \in B_\varepsilon(x_0) \cap A \text{ and } x \neq x_0$$

### Theorem 533 (Comparison criterion)

$f, g, h: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  and  $x_0 \in \mathbb{R}^n$  is an accumulation point of  $A$ . If:

$$1. \forall x \in A, g(x) \leq f(x) \leq h(x)$$

$$2. \lim_{x \rightarrow x_0} g(x) = \lim_{x \rightarrow x_0} h(x) = L$$

Then  $\lim_{x \rightarrow x_0} f(x) = L$

Eg.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^n \sin\left(\frac{1}{x}\right) \text{ with } x \neq 0 \text{ and } n \geq 1$$

$$\text{Show that } \lim_{x \rightarrow 0} x^n \sin\left(\frac{1}{x}\right) = 0$$

$$0 \leq \left| x^n \sin\left(\frac{1}{x}\right) \right| = |x^n| * \left| \sin\left(\frac{1}{x}\right) \right| \leq |x^n|$$

Therefore

$$\left| x^n \sin\left(\frac{1}{x}\right) \right| \leq |x^n|, \text{ that is } -|x^n| \leq x^n \sin\left(\frac{1}{x}\right) \leq |x^n|$$

$$\lim_{x \rightarrow 0} x^n \sin\left(\frac{1}{x}\right) = 0 \text{ by comparison criterion}$$

### Proposition 535

$f, g: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, x_0$  accumulation point of  $A$  and  $f(x) \rightarrow L \in \bar{\mathbb{R}}$  and  $g(x) \rightarrow H \in \bar{\mathbb{R}}$

1. If  $f(x) \geq g(x), \forall x \in B_\varepsilon(x_0)$  then  $L \geq H$
2. If  $L > H$  then a  $B_\varepsilon(x_0)$  exists such that  $f(x) > g(x)$

NB

$$\text{If } f(x) > g(x) \not\Rightarrow L > H$$

$$f, g: (0, +\infty) \rightarrow \mathbb{R}, f(x) = \frac{1}{x} \rightarrow 0 \text{ as } x \rightarrow +\infty$$

$$g(x) = 0 \rightarrow 0 \text{ as } x \rightarrow +\infty, \forall x \in (0, +\infty), f(x) > g(x) \text{ but } L = H = 0$$

$$\text{If } f(x) > g(x) \text{ in } B_\varepsilon(x_0) \Rightarrow L \geq H$$

### Proposition 536 (Algebra of limits)

$f, g: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, x_0$  accumulation point of  $A$  and  $f(x) \rightarrow L \in \bar{\mathbb{R}}$  and  $g(x) \rightarrow H \in \bar{\mathbb{R}}$

1.  $\lim_{x \rightarrow x_0} (f + g)(x) = L + H$  provided that  $L + H$  is not  $\pm \infty \mp \infty$
2.  $\lim_{x \rightarrow x_0} (f * g)(x) = L * H$  provided that  $L * H$  is not  $0 * (\pm \infty)$  or  $(\pm \infty) * 0$
3.  $\lim_{x \rightarrow x_0} \left(\frac{f}{g}\right)(x) = \frac{L}{H}$  provided that  $g(x) \neq 0$  in  $B_\varepsilon(x_0)$  with  $x \neq x_0$  and  $\frac{L}{H}$  is not  $\frac{\pm \infty}{\pm \infty}, \frac{0}{\pm \infty}$

Proof

1. Suppose that  $\{x_n\}$  is a sequence of point of  $A$ , with  $x_n \neq x_0, \forall n \geq 1$  s. t.  $x_n \rightarrow x_0$ .

Assume that  $f(x_n) \rightarrow L$  and  $g(x_n) \rightarrow H$ . Suppose that  $L + H$  is not an indefinite form, we have

$$(f + g)(x_n) = f(x_n) + g(x_n) \rightarrow L + H$$

$$\text{and } \lim_{x \rightarrow x_0} (f + g)(x) = L + H$$

### Proposition 538

$$\lim_{(x \rightarrow x_0)} f(x) = L \in \bar{\mathbb{R}}, \text{ with } L \neq 0, \quad \text{and } \lim_{(x \rightarrow x_0)} g(x) = 0$$

$\lim_{x \rightarrow x_0} \left(\frac{t}{g}\right)(x)$  exists  $\Leftrightarrow$  a  $V_\varepsilon(x_0)$  exists where  $g$  has a constant sign except at most at  $x_0$

1. If  $L > 0$  and  $g(x) \rightarrow 0^+$  or  $L < 0$  and  $g(x) \rightarrow 0^-$

$$\lim_{x \rightarrow x_0} \left(\frac{t}{g}\right)(x) = +\infty$$

2. If  $L > 0$  and  $g(x) \rightarrow 0^-$  or  $L < 0$  and  $g(x) \rightarrow 0^+$

$$\lim_{x \rightarrow x_0} \left(\frac{t}{g}\right)(x) = -\infty$$



## Limits of elementary functions

### 1. Power functions:

$$f: (0, +\infty) \rightarrow \mathbb{R}, f(x) = x^\alpha, (\alpha \in \mathbb{R})$$

$$x_0 \in (0, +\infty), \lim_{x \rightarrow x_0} x^\alpha = x_0^\alpha$$

Special case:

$$a < 0 \begin{cases} x^a \rightarrow +\infty, x \rightarrow 0^+ \\ x^a \rightarrow 0, x \rightarrow +\infty \end{cases}$$

Eg.

$$\lim_{x \rightarrow +\infty} \frac{1}{x^2} = 0$$

### 2. Exponential functions:

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = a^x (a > 0, a \neq 1)$$

$$\forall x_0 \in \mathbb{R}, \lim_{x \rightarrow x_0} a^x = a^{x_0}$$

$$a > 1, \begin{cases} a^x \rightarrow 0^+, x \rightarrow -\infty \\ a^x \rightarrow +\infty, x \rightarrow +\infty \end{cases}$$

$$0 < a < 1, \begin{cases} a^x \rightarrow 0^+, x \rightarrow +\infty \\ a^x \rightarrow +\infty, x \rightarrow -\infty \end{cases}$$

Eg.

$$\lim_{x \rightarrow +\infty} 2^x = +\infty$$

$$\lim_{x \rightarrow -\infty} 2^x = 0$$

### 3. Logarithmic functions:

$$f: (0, +\infty) \rightarrow \mathbb{R}, f(x) = \log_a x, a > 0, a \neq 1$$

$$\forall x_0 \in (0, +\infty), \lim_{x \rightarrow x_0} \log_a x = \log_a x_0$$

$$a > 1, \begin{cases} \log_a x \rightarrow -\infty, x \rightarrow 0^+ \\ \log_a x \rightarrow +\infty, x \rightarrow +\infty \end{cases}$$

$$0 < a < 1, \begin{cases} \log_a x \rightarrow -\infty, x \rightarrow 0^+ \\ \log_a x \rightarrow +\infty, x \rightarrow +\infty \end{cases}$$

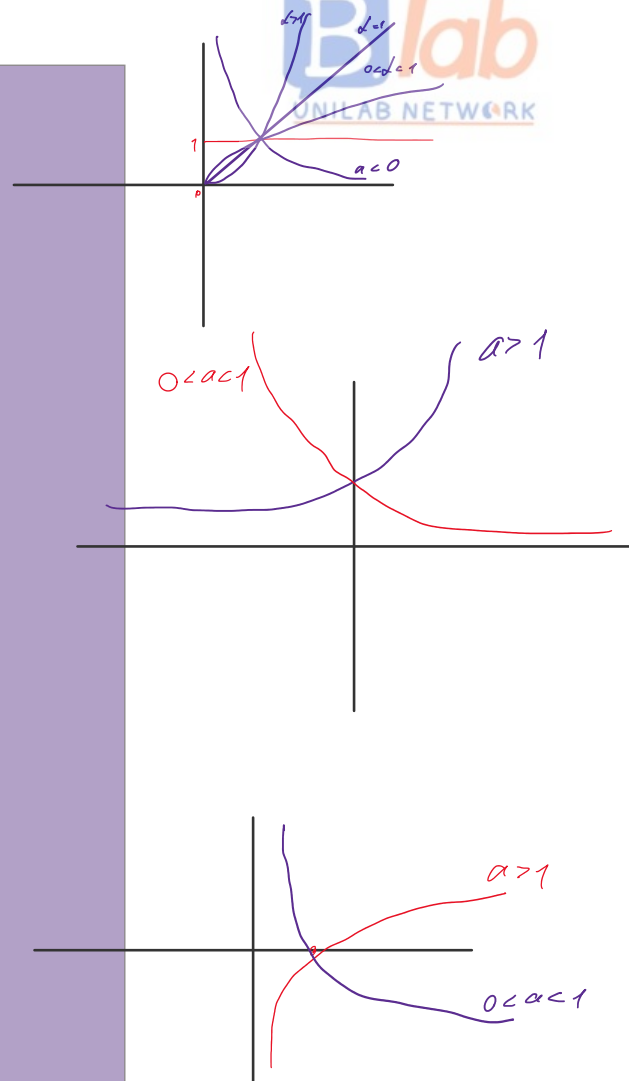
### 4. Elementary trigonometric functions:

$$f, g: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \sin(x), g(x) = \cos(x)$$

$$\forall x_0 \in \mathbb{R}, \lim_{x \rightarrow x_0} \sin x = \sin x, \lim_{x \rightarrow x_0} \cos x = \cos x_0$$

$$\nexists \lim_{x \rightarrow +\infty} \sin x$$

$$\nexists \lim_{x \rightarrow +\infty} \cos x$$



## Proposition 542

$$f, g: (0, +\infty) \rightarrow \mathbb{R} \text{ with } f(x) = \frac{\sin x}{x} \text{ and } g(x) = \frac{\cos x - 1}{x}$$

Then

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0^+} \frac{1 - \cos x}{x} = 0$$

$$\lim_{x \rightarrow 0^+} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

Proof

We know that

$$\forall x \in \left(0, \frac{\pi}{2}\right), 0 < \sin x < x < \tan x$$

We divide each term by  $\sin x > 0$  and we have

$$\forall x \in \left(0, \frac{\pi}{2}\right), 0 < \frac{\sin x}{x} < 1 < \frac{\tan x}{\sin x}$$

Therefore:

$$\forall x \in \left(0, \frac{\pi}{2}\right), 0 < \frac{\sin x}{x} < 1 < \frac{1}{\cos x} \left(\lim_{x \rightarrow 0^+} \cos x = 1\right)$$

Therefore:

$$0 < \frac{\sin x}{x} < 1 < \frac{1}{\cos x}, \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$$

We observe:

$$\frac{1 - \cos x}{x^2} = \frac{(1 - \cos x)(1 + \cos x)}{x^2(1 + \cos x)} = \frac{1 - \cos^2(x)}{x^2(1 + \cos x)} = \frac{\sin^2(x)}{x^2(1 + \cos x)} = \left(\frac{\sin^2 x}{x}\right)^2 * \frac{1}{1 + \cos x}$$

$$\lim_{x \rightarrow 0^+} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0^+} \left[ \left(\frac{\sin^2 x}{x}\right)^2 * \frac{1}{1 + \cos x} \right] = \frac{1}{2}$$

$$\lim_{x \rightarrow 0^+} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0^+} \frac{x(1 - \cos x)}{x^2} = \lim_{x \rightarrow 0^+} x \lim_{x \rightarrow 0^+} \frac{1 - \cos x}{x^2} = 0$$

NB

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0^+} \frac{1 - \cos x}{x} = 0, \quad \lim_{x \rightarrow 0^+} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

Eg.

$$\lim_{x \rightarrow 0^+} \frac{\cos x - 1}{x^2} = -\frac{1}{2}$$

$$\lim_{x \rightarrow 0^+} \frac{1 - \cos 2x}{4x^2} = \frac{1}{2} \quad (2x = t)$$

Important limits:

1. If  $f(x) \rightarrow \mp\infty$  as  $x \rightarrow x_0$  then

$$\lim_{x \rightarrow x_0} \left(1 + \frac{k}{f(x)}\right)^{f(x)} = e^k$$

NB

$$f(x) = x \rightarrow \pm\infty \text{ as } x \rightarrow \pm\infty, \quad \lim_{x \rightarrow \pm\infty} \left(1 + \frac{1}{k}\right)^k = e$$

Eg.

$$\lim_{x \rightarrow x_0} \left(1 - \frac{1}{2x^3}\right)^{x^3} = \frac{1}{\sqrt{e}}$$

2. If  $a > 0$  and  $f(x) > 0$  and  $f(x) \rightarrow 0$  as  $x \rightarrow x_0$

$$\lim_{x \rightarrow x_0} \frac{a^{f(x)} - 1}{f(x)} = \ln a$$

NB

$$\lim_{x \rightarrow x_0} \frac{e^x - 1}{x} = 1$$

EG.

$$\lim_{x \rightarrow x_0} \frac{e^{\frac{1}{x^2}} - 1}{\frac{1}{x^2}} = 1$$

$$\lim_{x \rightarrow x_0} \frac{e^{\frac{2}{x^2}} - 1}{\frac{1}{x^2}} = 2 \lim_{x \rightarrow x_0} \frac{e^{\frac{2}{x^2}} - 1}{\frac{2}{x^2}} = 2 * 1 = 2$$

$$\lim_{x \rightarrow x_0} \frac{e^{x^2+x} - 1}{x^2 + x} = 1$$

3. If  $a > 0, a \neq 1$  and  $f(x) \rightarrow 0$  as  $x \rightarrow x_0$  then

$$\lim_{x \rightarrow x_0} \frac{\log_a(1 + f(x))}{f(x)} = \frac{1}{\ln a}$$

NB.

$$\lim_{x \rightarrow x_0} \frac{\ln(1 + x)}{x} = 1$$

Eg.

$$\lim_{x \rightarrow x_0} \frac{\ln(1 + x^2 + 2x)}{x^2 + 2x} = 1$$

4. If  $f(x) \rightarrow 0$  as  $x \rightarrow x_0$

$$\lim_{x \rightarrow x_0} \frac{(1 + f(x))^a - 1}{f(x)} = a$$

NB

$$\lim_{x \rightarrow x_0} \frac{(1 + x)^a - 1}{x} = a$$

## Orders of convergence and divergence

Def.

$f, g: A \subseteq \mathbb{R} \rightarrow \mathbb{R}, x_0 \in \bar{A}$  accumulation point of  $A$

s.t.

$\exists B_\varepsilon(x_0)$  where  $g(x) \neq 0$  for any  $x \in A \cap B_\varepsilon(x_0)$

$$1. \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0 \Rightarrow f \text{ is negligible w.r.t. } x$$

$\rightarrow x_0 (f = o(g) \text{ as } x \rightarrow x_0)$

$$2. \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = k \neq 0 \Rightarrow f \text{ and } g \text{ are comparable as } x$$

$\rightarrow x_0 (f \asymp g \text{ as } x \rightarrow x_0)$

NB

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 1 \Rightarrow f, g \text{ are asymptotic } (f \sim g)$$

Eg.

$$f, g: (0, +\infty) \rightarrow \mathbb{R}, f(x) = \frac{1}{x}, g(x) = \frac{1}{x^2}$$

1.  $x \rightarrow 0$

$$\lim_{x \rightarrow 0} \frac{1/x}{1/x^2} = \lim_{x \rightarrow 0} x = 0 \Rightarrow f(x) = o(g(x))$$

2.  $x \rightarrow +\infty$

$$\lim_{x \rightarrow 0} \frac{1/x}{1/x^2} = \lim_{x \rightarrow 0} x = +\infty \Rightarrow g(x) = o(f(x))$$

Eg.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \Leftrightarrow \sin x \sim x \text{ as } x \rightarrow 0$$

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 \Leftrightarrow \ln(1+x) \sim x$$

Eg

$$f, g: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 + x, g(x) = x + 1$$

$$\lim_{x \rightarrow 0} \frac{x^2 + x}{x + 1} = 0 \Rightarrow x^2 + x = o(x + 1)$$

Special properties:

1. The relation of comparability  $\asymp (\sim)$  is reflexive, symmetric and transitive
2. The relation of negligibility is transitive

### **Proposition 544**

$f, g: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$  and  $c \neq 0$

1.  $o(f) = o(f) = o(f)$
2.  $o(f) * o(g) = o(f * g)$
3.  $c * o(f) = o(f)$
4.  $o(g) + o(f) = o(f)$  if  $g = o(f)$

### **Proposition 546**

$K, h > 0, \alpha > 1$  and  $d > 1$

1.  $x^k = o(\alpha^x)$  as  $x \rightarrow +\infty$
2.  $x^h = o(x^k)$  if  $h < k$  as  $x \rightarrow +\infty$
3.  $\log_a x = o(x^k)$  as  $x \rightarrow +\infty$

## **Asymptotic equivalence**

If  $f(x) \sim g(x)$  as  $x \rightarrow x_0$  then

$$\lim_{x \rightarrow x_0} f(x) = L \in \overline{\mathbb{R}} \Leftrightarrow \lim_{x \rightarrow x_0} g(x) = H \in \overline{\mathbb{R}}$$

Eg.

$$e^{x^2} - 1 \sim x^2, x \rightarrow 0 \left( \lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{x^2} = 1 \right)$$

$$\lim_{x \rightarrow 0} e^{x^2} - 1 \sim \lim_{x \rightarrow 0} x^2$$

### **Lemma 547**

$f(x) \sim g(x)$  and  $h(x) \sim l(x)$  as  $x \rightarrow x_0$

1.  $f(x) * h(x) \sim g(x) * l(x)$
2.  $\frac{f(x)}{h(x)} \sim \frac{g(x)}{l(x)}, x \rightarrow x_0$  provided that  $h(x), l(x) \neq 0$

### **Lemma 548**

$f(x) \sim f(x) + o[f(x)]$  as  $x \rightarrow x_0$

Therefore

$$\lim_{x \rightarrow x_0} f(x) = L \Leftrightarrow \lim_{x \rightarrow x_0} [f(x) + o[f(x)]] = L$$

Eg.

$$\left\{ \lim_{x \rightarrow 0} \frac{x + x^2}{\sin x} = \lim_{x \rightarrow 0} \frac{x + o(x)}{\sin x} = \lim_{x \rightarrow 0} \frac{x}{\sin x} = 1 \right.$$

NB

We have:

$$1. [f(x) + o[f(x)]] * [g(x) + o[g(x)]] \sim f(x) * g(x) \text{ as } x \rightarrow x_0$$

Eg:  $(x^2 + x) * (x^3 + x^2 + x) = [x + o(x)][x + o(x)] \sim x^2 \text{ as } x \rightarrow 0$

$$2. \frac{f(x) + o(f)}{g(x) + o(g)} \sim \frac{f(x)}{g(x)} \text{ as } x \rightarrow x_0$$

Eg:  $\frac{x^2+x}{x^3+x^2+x} = \frac{x+o(x)}{x+o(x)} \sim \frac{x}{x} = 1$

If  $f(x) \rightarrow 0$  as  $x \rightarrow x_0$  then  $f$  is called an infinitesimal function as  $x \rightarrow x_0$

Eg.

$$\begin{cases} \sin x \rightarrow 0 \\ x \rightarrow 0 \end{cases}$$

If  $f(x) \rightarrow 0$  as  $x \rightarrow x_0$  then  $f$  is called an infinite function as  $x \rightarrow x_0$

EG.

$$\begin{cases} \frac{1}{e^x} \rightarrow +\infty \\ x \rightarrow -\infty \end{cases}$$

1. If  $f, g$  are infinitesimal functions as  $x \rightarrow x_0$  ( $f, g \rightarrow 0$ )

$$\begin{cases} f = o(g) \Leftrightarrow \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0 \\ x \rightarrow x_0 \end{cases} \quad f \text{ is an infinitesimal function of higher order w.r.t. to } g(x), x \rightarrow x_0$$

Eg.

$f, g: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^3 + x^2, g(x) = x$  as  $x \rightarrow 0, f, g$  are infinitesimal functions

$$\lim_{x \rightarrow 0} \frac{x^3 + x^2}{x} = \lim_{x \rightarrow 0} x^2 + x = 0$$

$$x^3 + x^2 = o(x)$$

2.  $f, g$  infinite functions as  $x \rightarrow x_0$  ( $f, g \rightarrow \pm\infty$  as  $x \rightarrow x_0$ )

$$\begin{cases} f = o(g) \text{ as } x \rightarrow x_0 \Leftrightarrow \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0 \\ f \text{ is an infinite of a lower order w.r.t. } g \text{ as } x \rightarrow x_0 \end{cases}$$

EG.

$$f, g: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^3, g(x) = e^x, \frac{x^3}{e^x} \rightarrow 0$$

$$x^3 = o(e^3) \text{ as } x \rightarrow +\infty$$

## Scales of infinitesimal and infinite functions

1. Infinite functions:

$$\begin{matrix} f(x) = a^x, a > 1 \\ g(x) = x^k, k > 0 \\ z(k) = (\log_a x)^k, k > 0 \end{matrix} \Rightarrow \begin{cases} \frac{x^k}{a^x} \rightarrow 0 \\ \frac{(\log_a x)^k}{x^k} \rightarrow 0 \\ \frac{(\log_a x)^k}{e^x} \rightarrow 0 \end{cases}$$

2. Infinitesimal functions:

$$\begin{array}{l} f(x) = a^x, 0 < a < 1 \\ g(x) = x^k, k < 0 \\ z(k) = (\log_a x)^k, k < 0 \end{array} \Rightarrow \begin{cases} \frac{x^k}{a^x} \rightarrow 0 \\ \frac{(\log_a x)^k}{x^k} \rightarrow 0 \\ \frac{(\log_a x)^k}{e^x} \rightarrow 0 \end{cases}$$

NB

$A \subseteq \mathbb{R}^n, x^*$  can be  $\begin{cases} \text{an isolated point of } A \\ \text{an accumulation point of } A \end{cases}$

Eg.  $(2,10] \cup \{20\}$

$x^* = 20 \in A, \exists B_\epsilon(20): B_\epsilon(20) \cap A$

$x^* = 5$  interior point of  $A \Rightarrow x^* = 5$  is an accumulation point  $= \{20\}$

$\hat{x} = 2 \notin A$  is an acc. point of  $A$

Definition

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  and  $x_0 \in A$  an accumulation point.

$f$  is **continuous** function at  $x_0$  if

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

By convention,  $f$  **continuous** at any isolated point of  $A$ .

Eg.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

Is  $f$  a constant function at  $x_0 = 0$

$x_0 = 0 \in \mathbb{R}$  and  $x_0 = 0$  is an accumulation point of  $\mathbb{R}$ .

$$\begin{cases} \lim_{x \rightarrow 0^-} f(x) = +\infty \\ \lim_{x \rightarrow 0^+} f(x) = +\infty \end{cases} \Leftrightarrow \lim_{x \rightarrow 0} f(x) = +\infty$$

$$f(0) = 2$$

$$\lim_{x \rightarrow 0} f(x) = +\infty \neq 2 = f(0), f \text{ is not continuous}$$

Eg.

$$g: \mathbb{R} - \{0\} \rightarrow \mathbb{R}, g(x) = \frac{1}{x^3}$$

Is  $g$  continuous on any  $x_0 \in \mathbb{R} - \{0\}$

Yes, it is continuous at each point of its natural domain

$$x_0 \in \mathbb{R} - \{0\}, \lim_{x \rightarrow x_0} \frac{1}{x^3} = \frac{1}{x_0^3} = f(x_0)$$

$f$  is continuous on  $x_0 \in \mathbb{R} - \{0\} \Rightarrow f$  is continuous on  $\mathbb{R} - \{0\}$

### Proposition 552 (Sequential characterization of continuity)

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous at  $x_0 \in \mathbb{R} \Leftrightarrow x_n \rightarrow x_0 \Rightarrow f(x_n) \rightarrow f(x_0)$

$\forall \{x_n\}$  of points of  $A$

Eg.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\forall x_0 \in \mathbb{R} - \{0\}, f$  is continuous at  $x_0 \lim_{x \rightarrow x_0} \sin\left(\frac{1}{x}\right) = \sin\left(\frac{1}{x_0}\right) = f(x_0)$

Is  $f$  continuous at  $x_0 = 0$ ? No!

NB

$\{x_n\}, \{y_n\}$  sequences of points of  $A$  such that  $x_n \rightarrow 0, y_n \rightarrow 0$

If  $f(x_n) \rightarrow L_1$  and  $f(y_n) \rightarrow L_2, L_1 \neq L_2$  then  $f$  is not continuous at  $x_0 = 0$

$$\begin{cases} x_n = \frac{1}{2n\pi}, y_n = \frac{1}{2n\pi + \frac{\pi}{2}} \\ x_n \rightarrow 0, y_n \rightarrow 0 \text{ as } n \rightarrow +\infty \end{cases}$$

$$f(x_n) = \sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{1}{\frac{1}{2n\pi}}\right) = \sin(2n\pi) = 0 \rightarrow 0 = L_1$$

$$f(y_n) = \sin\left(\frac{1}{y_n}\right) = \sin\left(\frac{1}{\frac{1}{2n\pi + \frac{\pi}{2}}}\right) = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1 = L_2$$

$L_1 \neq L_2 \Rightarrow f$  is not continuous on  $x_0 = 0$

EG.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} \frac{\ln(1+x)}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

$$g: \mathbb{R}^2 \rightarrow \mathbb{R}, g(x) = \begin{cases} \frac{e^{x_1^2+x_2^2} - 1}{x_1^2 + x_2^2}, & (x_1, x_2) \neq (0,0) \\ b, & (x_1, x_2) = (0,0) \end{cases}$$

$$\forall x_0 \in \mathbb{R} - \{0\}, \lim_{x \rightarrow x_0} \frac{\ln(1+x)}{x} = \frac{\ln(1+x_0)}{x_0} = f(x_0)$$

$$x_0 = 0, \lim_{x \rightarrow x_0} \frac{\ln(1+x)}{x} = 1, f(0) = a \begin{cases} = 0 \\ \neq 0 \end{cases}$$

$$\forall x^0 \in \mathbb{R}^2 - \{(0,0)\}, \lim_{x \rightarrow x_0} \frac{e^{x_1^2+x_2^2} - 1}{x_1^2 + x_2^2} = \frac{e^{x_1'^2+x_2'^2} - 1}{x_1'^2 + x_2'^2} = f(x^0)$$

$$\lim_{x \rightarrow (0,0)} \frac{e^{x_1^2+x_2^2} - 1}{x_1^2 + x_2^2} = 1, f(0) = b \begin{cases} b = 1 \Rightarrow f \text{ continuous at } x = 0 \Rightarrow f \text{ continuous on } \mathbb{R}^2 \\ b \neq 1 \Rightarrow f \text{ not continuous at } x = 0 \Rightarrow f \text{ not continuous on } \mathbb{R}^2 \end{cases}$$

**All the elementary functions are continuous on their natural domain!**

EG.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} e^{2x}, & x \leq 0 \\ x + a, & x > 0 \end{cases}$$

$$\begin{cases} \lim_{x \rightarrow 0^+} e^{2x} = 1 \\ \lim_{x \rightarrow 0^-} x + a = a \end{cases} \Rightarrow \exists \lim_{x \rightarrow 0} f(x) \text{ exists} \Leftrightarrow a = 1$$

Eg.

$$g: \mathbb{R}^3 \rightarrow \mathbb{R}, g(x) = x_1^2 + 2x_2^2 + \log_7(x_3^2 + 1)$$

Is  $g(x)$  continuous on  $\mathbb{R}^3$ ?

$g(x)$  is separable function, each part is continuous, hence their sum is also continuous

NB

A separable function  $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  given by

$$f(x) = \sum_{i=1}^n g_i(x_i)$$

Is continuous at  $x_0 \in A$  if each function  $g_i: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous at  $x_0 \in R$

**Proposition 561**

The norm  $\|\cdot\|: \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous on  $\mathbb{R}^n$

Proof

NB

$$\forall a, b \in \mathbb{R}^n, \left| \|a\| - \|b\| \right| \leq \|a - b\|$$

Consider sequence of points of  $\mathbb{R}^n$  s.t.  $x_n \rightarrow x_0 \in \mathbb{R}^n$

From the norm inequality:

$$0 \leq \left| \|x_n\| - \|x_0\| \right| \leq \|x_n - x_0\|$$

But when  $x_n \rightarrow x_0$  we have  $\|x_n - x_0\| \rightarrow 0$  and

$$0 \leq \left| \|x_n\| - \|x_0\| \right| \leq 0 \Rightarrow \left| \|x_n\| - \|x_0\| \right| \rightarrow 0 \Rightarrow \|x_n\| \rightarrow \|x_0\|$$

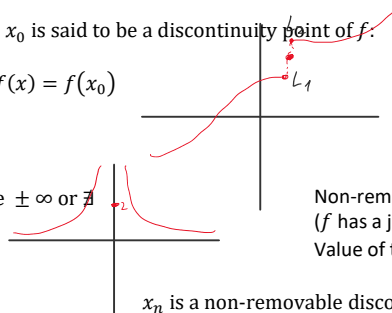
therefore  $\|\cdot\|: \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous for all  $\mathbb{R}^n$

If a function  $f$  is not continuous at  $x_0 \in \mathbb{R}$ , then  $x_0$  is said to be a discontinuity point of  $f$ .

We move from the definition of continuity:

$$\lim_{x \rightarrow x_0} f(x) = f(x_0) \Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0^+} f(x) = f(x_0)$$

- $\lim_{x \rightarrow x_0} f(x) = L \neq L' = f(x_0)$
- $\lim_{x \rightarrow x_0^-} f(x) = L_1 \neq L_2 = \lim_{x \rightarrow x_0^+} f(x)$
- Either  $\lim_{x \rightarrow x_0^-} f(x)$  or  $\lim_{x \rightarrow x_0^+} f(x)$  are  $\pm \infty$  or  $\nexists$



Non-removable discontinuity of  $f$  ( $f$  has a jump at  $x_0$ ).

$$\text{Value of the jump is } S(x_0) = \lim_{x \rightarrow x_0^+} f(x) - \lim_{x \rightarrow x_0^-} f(x)$$

$x_n$  is a non-removable discontinuity point

$x_0$  is a removable discontinuity point

EG.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ -2, & x = 0 \end{cases}$$

$\nexists \lim_{x \rightarrow 0} f(x)$ ,  $x_0 = 2$  is a discontinuity point of  $f$

Eg.

$$g: \mathbb{R}^2 \rightarrow \mathbb{R}, g(x) = \begin{cases} x_1^2 + x_2^2 & \text{if } x \neq (0,0) \\ 2 & \text{if } x = (0,0) \end{cases}$$

$g$  is continuous on  $\forall x \in \mathbb{R}^2 - \{(0,0)\}$

$$\lim_{x \rightarrow 0} (x_1^2 + x_2^2) = 0 \neq 2 = f(0,0)$$

**Proposition 566**

$f, g: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  continuous at  $x_0 \in A$ . Then

1.  $f + g$  is continuous at  $x_0$
2.  $f * g$  is continuous at  $x_0$
3.  $\frac{f}{g}$  is continuous at  $x_0$  provided that  $g(x_0) \neq 0$

Proof

Assume that  $f, g$  are continuous at  $x_0 \in A$

$$\lim_{x \rightarrow x_0} f(x) = f(x_0) \text{ and } \lim_{x \rightarrow x_0} g(x) = g(x_0)$$

We have

$$\begin{aligned} \lim_{x \rightarrow x_0} (f + g)(x) &= \lim_{x \rightarrow x_0} [f(x) + g(x)] = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x) \\ &= f(x_0) + g(x_0) = (f + g)(x_0) \end{aligned}$$

Hence  $f + g$  is continuous at  $x_0 \in A$

**Same strategy for the other two**

NB

1.  $f$  is continuous on  $[a, b]$  when
  - $f$  cont. in  $(a, b)$ ,  $f$  is cont. from the right of  $a$ 

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$
  - And  $f$  is continuous from the left at  $b$ 

$$\lim_{x \rightarrow b^-} f(x) = f(b)$$
2.  $f \in C(E)$  with  $E \subseteq A \subseteq \mathbb{R}^n$   
 $f$  belongs to the class of all the continuous functions on  $E$

EG.

$$\begin{aligned} f: \mathbb{R} \rightarrow \mathbb{R}, f(x) &= e^x \\ g: (0, +\infty) \rightarrow \mathbb{R}, g(x) &= \ln x \\ f &\in C(E) \end{aligned}$$

## Continuity of composite functions

Proposition

$$f: A_f \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, \text{ and } g: A_g \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \text{ and } f(A_f) \subseteq A_g$$

If  $f$  is continuous at  $x_0 \in A_f$  and  $g$  is continuous at  $f(x_0) \in A_g$  then  
 $g \circ f$  is continuous at  $x_0 \in A_f$

Proof

Consider  $\{x_n\}$  of points of  $A$  s.t.  $x_n \rightarrow x_0$

Assume that  $f(x_n) \rightarrow f(x_0)$  and  $g[f(x_n)] = g \circ f(x_n)$

Therefore  $g \circ f$  is continuous at  $x_0 \in A_f$



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For doubts or suggestions on the handout:



**Dzianis Rabchuk**



+39 328 705 1367



@\_dzianis\_ka

For info about our Teaching Division:



**PIETRO VILLA**



+39 346 2100003



@pietro\_villa\_\_



**CHIARA TUA**



+39 347 9789059



@chiara\_tua

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