UNILAB NETWORK

MATHEMATICS MODULE 1 (II PARTIAL)

NOTES

A.Y. 2023 - 2024

Written by Dzianis Rabchuk

This handout has been written by students with no intention to substitute the University official materials. Its purpose is to be an instrument useful to the exam preparation, but it does not give a total knowledge about the program of the course it is related to, as the materials of the university website or professors

Chapter 15 - Linear functions and operators

mercoledì 18 ottobre 2023 08:34

A $m \times n$ matrix is a table of real numbers with m rows and n columns

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The generic element or entry of a matrix is indicated with $[a_{ij}]$ $i = 1, \dots, n; j = 1, \dots, m$

$A_{m \times n}$ = Matrix with n rows and n columns

Eg.

$$A_{2\times 3} = \begin{bmatrix} 2 & 3 & -1 \\ \pi & -2 & 2 \end{bmatrix}$$

NB

If $m = n$ then the $n \times n$ is called a square matrix of order n

Eg.

$$D = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 2 & -1 \\ 4 & 440 & 3 \end{bmatrix} \text{ Square matrix of order 3}$$

Remarkable matrices:

1. Identity or unit matrix I_n

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \dots & \cdots \\ \vdots & ⬚ & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix},$$

Eg.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. Zero or Null matrix with dimenstion $m \times n$

$$O_{m\times n} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

$M(m, n)$ − Set of all the $m \times n$ matrices of real numbers

Operations with matrices:

1. Addition of matrices, $A, B \in M(m, n)$

$$A + B = \left[a_{ij} + b_{ij}\right]$$

2. Scalar multiplication $A \in M(m, n), \alpha \in R$

$$\alpha A = \left[\alpha a_{ij}\right]$$

EG.

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & -2 \end{bmatrix}$$

$$2A = 2 * \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 6 \\ 0 & 2 & -2 \end{bmatrix}$$

$$(-1)B = (-1)\begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

Properties:

$A, B, C \in M(m, n)$ and $d, c \in R$

1. $\boldsymbol{A + B = B + A}$
2. $\boldsymbol{(A + B) + C = A + (B + C)}$
3. $\boldsymbol{A + 0 = 0 + A = A}$ $\left(0 \textit{ is called a neutral element for the addition}\right)$
4. $\boldsymbol{A + (-A) = (-A) + A = 0}$ $\left(A \textit{ is called to be the opposite of } A\right)$
5. $\boldsymbol{d(A + B) = dA + dB}$

6. $(\boldsymbol{d}+\boldsymbol{c})\boldsymbol{A} = \boldsymbol{dA} + \boldsymbol{cA}$
7. $\boldsymbol{1} * \boldsymbol{A} = \boldsymbol{A} * \boldsymbol{1} = \boldsymbol{A}$ $(1\ \textit{is called a neutral element of the scalar multiplication})$
8. $\boldsymbol{d}(\boldsymbol{cA}) = (\boldsymbol{dc})\boldsymbol{A}$

Proof

1. Assume $A, B \in M(m,n)$. We have

$$A + B = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A$$

Therefore, $A + B = B + A$

2. Assume that $A, B, C \in M(m,n)$

$$A + (B + C) = [a_{ij} + (b_{ij} + c_{ij})] = [a_{ij} + b_{ij} + c_{ij}] = [(a_{ij} + b_{ij}) + c_{ij}] = (A + B) + C$$

Therefore, $A + (B + C) = (A + B) + C$

3.

Same strategy for the rest

NB

1. $M(m,n)$ Set of all matrices with dimension $m \times n$
2. Two operations:
   a. Addition
   b. Scalar multiplication
3. 8 properties (4 for addition + 4 for scalar multiplication)

$M(m,n)$ = Vector space

EG.

$$V = \left\{A \in M(2): \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}, a_{11}, a_{22} \in R\right\}$$

Show that V is a linear subspace of $M(2)$

1. $V \neq 0, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in V$

   $0_2 \in V?, a_{11}, a_{22} = 0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$

2. $A + B = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & 0 \\ 0 & a_{22+b_{22}} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ 0 & c_{22} \end{bmatrix} \in V$ – Closed w.r.t. addition

3. $A \in V, d \in R$, hence $dA \in M(2)$

$$dA = d\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} d * a_{11} & 0 \\ 0 & d * a_{22} \end{bmatrix} = \begin{bmatrix} t_{11} & 0 \\ 0 & t_{22} \end{bmatrix} \in V$$ – Closed w.r.t. scalar multiplication

Hence, V is a vector subspace of M(2)

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow B_V = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

NB.

$A, B \in M(m,n)$; A, B are said to equal matrices

$$(A = B) \Leftrightarrow \begin{cases} A, B \text{ have the same dimension} \\ \forall i, j, a_{ij} = b_{ij} \end{cases}$$

Eg.

$$A = \begin{bmatrix} a^2 & 1 \\ -1 & b^2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, A = B \Leftrightarrow \begin{cases} a^2 = 1 \\ b^2 = 0 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 0 \end{cases}$$

Special square matrices

1. Symmetric matrix:

   $\forall ij, a_{ij} = a_{ji}$

   Eg.

$$\begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$$

2. Lower triangular matrix:

   $a_{ij} = 0$ if $i < j$

   Eg.

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 2 & 6 & 0 \\ 1 & 3 & -9 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 2 & 6 & 0 \\ 1 & 3 & -9 \end{bmatrix}$$

3. Upper triangle matrix:

   $a_{ij} = 0$ if $i > j$

   Eg.

$$\begin{bmatrix} 5 & 6 & 7 \\ 0 & 4 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

4. Diagonal matrix:

   $a_{ij} = 0$ if $i \neq j$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Eg.

$$A_{2\times 3} = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & -1 \end{bmatrix}$$

$$A^T_{3\times 2} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}$$

Definition

Let A be a $m \times n$ matrix. The $n \times m$ matrix (A^T) given by:

$$A^T = [a_{ji}]$$

Is called the transposed matrix of A

Properties:

$A, B \in M(m,n)$ and $\alpha \in R$

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(\alpha A)^T = \alpha * A^T$

NB.

$A, B \in M(m,n), \alpha, \beta \in R$

$(\alpha A + \beta B)^T = \alpha A^T + \beta B^T$

Let A and B be matrices with dimensions, respectively $m \times n$ and $p \times q$

The multiplication of A and B (AB) is possible $\Leftrightarrow n = p$ (if $n \neq p$ then $\nexists AB$)

The product matrix $A_{m\times n} * B_{p\times q} = C_{m\times q}$

Eg.

$A_{2\times 2} * B_{2\times 3} = C_{2\times 3}$

$\nexists B_{2\times 3} * A_{2\times 2}$

The generic element of the product matrix AB is equal to

$$AB = \left[\sum_{k=1}^{n} a_{ik} * b_{kj}\right] = a^i * b^j$$

Eg.

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} * \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2*2+1*(-1) & 2*(-1)+1*0 & 2*0+1*2 \\ (-1)*2+(-1)*3 & (-1)(-1)*0*3 & (-1)*0+3*2 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 2 \\ -5 & 1 & 6 \end{bmatrix}$$

Eg

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 2 & -1 \\ -2 & 10 \end{bmatrix}$$

Properties:

$A, B, C \in M(m,n)$, Assume that all products exist

1. $(AB)C = A(BC)$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $\alpha(AB) = (\alpha A) * B = A * (\alpha B)$
5. $(AB)^T = B^T * A^T$

Inverse matrices

Def

Let A be a **square matrix** of order n. Matrix A is said to be invertible if a square matrix of order n B exists s.t.

$A * B = B * A = I_n$

B is called an **inverse matrix** of A

A^{-1} inverse matrix of A

$A^{-1} * A = A * A^{-1} = I_n$

Eg.

$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ – is A invertible?

$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$AB = I_2$

$$AB = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} * \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a + c & 2b + d \\ 3c & 3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{cases} 2a + c = 1 \\ 2b + d = 0 \\ 3c = 0 \\ 3d = 1 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{2} \\ b = -\frac{1}{6} \\ c = 0 \\ d = \frac{1}{3} \end{cases}$$

$$B = A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} * \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 2 * \frac{1}{2} + 1 * 0 & 2 * \left(-\frac{1}{6}\right) + \frac{1}{3} * 1 \\ \frac{1}{2} * 0 + 3 * 0 & \left(-\frac{1}{6}\right) * 0 + \frac{1}{3} * 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Properties of A,B square matrices:

1. $(A^{-1})^{-1} = A$
2. $(A^T)^{-1} = (A^{-1})^T$
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $\forall \alpha \in R - \{0\}, (\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$

Eg.

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ Is A invertible

Suppose that B exists s.t. $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, AB = I_2$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} * \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + 2c & b + 2d \\ 2a + 4c & 2b + 4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{cases} a + 2c = 1 \\ b + 2d = 0 \\ 2a + 4c = 0 \\ 2b + 4d = 1 \end{cases}$$ – Impossible, hence $\nexists A^{-1}$

Determinant of a square matrix of order n

Def

Let A be a square matrix of order n. The determinant of A is a function

$\det: M(n) \to R$

Such that

1. If n=1 then $\det(A) = a_{11}$
2. If $n > 1$ then $\det(A) = \sum_{j=1}^{n} (-1)^{1+j} * a_{1j} * \det[A_{1j}]$

Where a_{1j} is a matrix obtained from A eliminating the first row and the j-th column of A

Eg.

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} * a_{1j} * \det[A_{1j}]$$

$$\det(A) = (-1)^{1+1} * a_{11} * \det(A_{11}) + (-1)^{1+2} * a_{12} * \det[A_{21}] = (-1)^2 * 2 * \det[3] + (-1)^3 * \det[-1]$$

$$= 1 * 2 * 3 + (-1) * 1 * (-1) = 6 + 1 = 7$$

NB

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \det(A) = a_{11} * a_{22} - a_{12} * a_{21}$$

Eg.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ -1 & 2 & 1 \end{bmatrix}, \text{ find the } \det[A]$$

$$\det(A) = \sum_{j=1}^{3} (-1)^{1+j} * a_{1j} \det(A_{1j})$$

$$= (-1)^{1+1} * a_{11} \det \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} + (-1)^{1+2} * a_{12} \det \begin{bmatrix} 0 & 3 \\ -1 & 1 \end{bmatrix} + (-1)^{1+3} * a_{13} \det \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= 1 * 1 * (1 - 6) - 1 * 2(0 + 3) + 1 * (-1) * (0 + 1) = -5 - 6 - 1 = -12$$

NB.

A is called a **singular matrix** $\Leftrightarrow \det(A) = 0$

A is called a **non-singular matrix** $\Leftrightarrow \det(A) \neq 0$

Eg.

$$A = \begin{bmatrix} d & 1 \\ 1 & d \end{bmatrix}, d \in R$$

$\det(A) = d^2 - 1, \det(A) = 0, d^2 - 1 = 0 \Rightarrow d = \pm 1$

- A is singular matrix if $d = \pm 1$
- A is not singular matrix if $d \in R - \{-1\} - \{1\}$

Properties:

1. $\det(I_n) = 1$
2. $\det(A^T) = \det(A)$
3. $\det(A * B) = \det(A) * \det(B)$, assume that AB exists

Type equation here.

Continuity

giovedì 2 novembre 2023 14:48

Theorem (Bolzano)

Let $f: [a, b] \to R$ be a continuous function

If

$f(a) \cdot f(b) \leq 0$

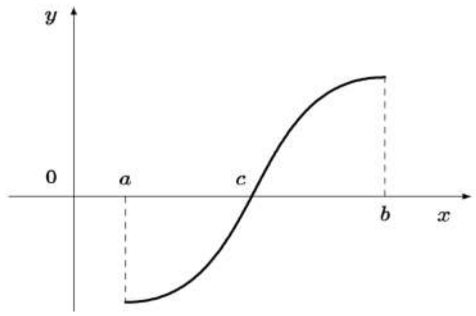
Then $\exists c \in [a, b]$ such that

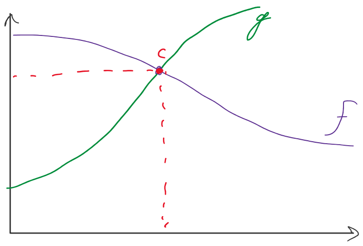
$f(c) = 0$

Moreover, if f is strictly monotone such c is unique

Meaning that $f(a)$ & $f(b)$ are of different sign

Graphically:





Proposition

Let $f, g: [a, b] \to R$ be continuous. If $f(a) \geq g(a)$ and $f(b) \leq g(b), \exists c \in [a, b]$ s.t.

$f(c) = g(c)$

If f is strictly decreasing and g is strictly increasing, such c is unique.

Proof

Set $f(a) \geq g(a)$ and $f(b) \leq f(b)$

Define an auxiliar function $h: [a, b] \to R$

Defined by

$h = f - g$

By hypothesis

$h(a) \geq 0 \wedge h(b) \leq 0$

Since h is continuous, by Bolzano theorem, $\exists c \in [a, b]$ s.t. $h(c) = 0$

$\Rightarrow h(c) = f(c) - g(c) = 0$

$\Rightarrow f(c) - g(c) = 0$

$\Rightarrow f(c) = g(c)$

QED

Application

- Consider a market M of some agricultural good, say potatoes
- A demand function $D : [a, b] \to \mathbb{R}$ and by a supply function $S : [a, b] \to \mathbb{R}$, with $a \geq 0$
- $D(p)$ is the overall amount of potatoes demanded at price p by consumers
- $S(p)$ is the overall amount of potatoes supplied at price p by producers
- Both respond instantaneously to changes in the market price p (e.g., producers are able to adjust in real time their production levels according to the market price p)

Definition: means that $p \in [a, b]$, $q \in R_+ (\Rightarrow q \geq 0)$

A pair $(p, q) \in [a, b] \times R_+$ of prices and quantities is called an **equilibrium** of market M if

$q = D(p) = S(p)$

The pair

(p, q)

is the equilibrium of our market

Example (not only exists, but also is unique)

- For instance, consider linear demand and supply functions:

$$D(p) = \alpha - \beta p$$
$$S(p) = \gamma p$$

with $\alpha, \beta, \gamma > 0$

- Consider the interval

$$[a, b] = \left[0, \frac{\alpha}{\beta}\right]$$

where both demand and supply are positive

- For our linear economy, the equilibrium condition becomes

$$\alpha - \beta p = \gamma p$$

- The equilibrium price and quantity are

$$p = \frac{\alpha}{\beta + \gamma}$$

and

$$q = D(p) = \alpha - \beta p = \alpha - \beta \frac{\alpha}{\beta + \gamma} = \frac{\alpha\gamma}{\beta + \gamma}$$

- The pair

$$\left(\frac{\alpha}{\beta + \gamma}, \frac{\alpha\gamma}{\beta + \gamma}\right)$$

is the equilibrium of our market of potatoes

Proposition

Let $D: [a, b] \to R$ and $S: [a, b] \to R$ be continuous and such that

$D(a) \geq S(a)$ and $D(b) \leq S(b)$

Then there exists a market equilibrium

$(p, q) \in [a, b] \times R_+$

If, in addition, D is strictly decreasing and S is strictly increasing, such equilibrium is unique

The Weierstrass Theorem (a baby version)

A continuous function $f: [a, b] \to R$ has at least one minimizer and at least one maximizer in $[a, b]$, that is

$\exists x_1, x_2 \in [a, b]$

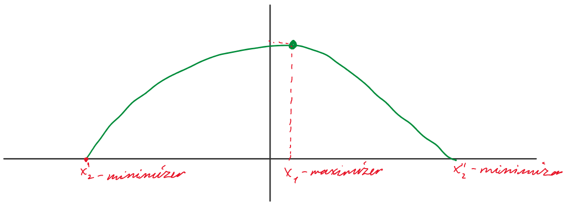
s.t.

$$f(x_1) = \max_{x \in [a,b]} f(x)$$

and

$$f(x_2) = \min_{x \in [a,b]} f(x)$$

Key conditions:
- Continuity
- Closeness



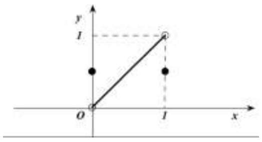
Eg.

Define $f: [0,1] \to R$

$$f(x) = \begin{cases} x \text{ if } x \in (0,1) \\ \frac{1}{2} \text{ if } x \in \{0,1\} \end{cases}$$

The function is defined on a compact interval, but isn't continuous

It has no maximizers and minimizers



Intermediate Value Theorem

Let $f: [a, b] \to R$ be continuous. Set

$$m = \min_{x \in [a,b]} f(x)$$

and

$$M = \max_{x \in [a,b]} f(x)$$

Then, $\forall z$ s.t. $m \leq z \leq M$

$\exists c \in [a, b]$ s.t. $f(c) = z$

NB:

f strictly monotonic $\Rightarrow c$ is unique

Proof

Instrumental lemma

Let $f: [a, b] \to R$ be continuous, with $f(a) \leq f(b)$.

If $f(a) \leq z \leq f(b)$ then $\exists c \in [a, b]$ s.t. $f(c) = z$. If f is strictly increasing, such c is unique

Proof of the lemma

Let $f(a) \leq f(b)$. Define an auxiliar function

$h: [a, b] \to R$ by $h(x) = f(x) - z$

we have

$h(a) = f(a) - z \leq 0$

$h(b) = f(b) - z \geq 0$

Since h is continuous, by Bolzano's theorem

$\exists c \in [a, b]$ s. t. $h(c) = 0$

Hence $f(c) = z$

Proof of intermediate

Set

$$m = \min_{x\in[a,b]} f(x)$$

and

$$M = \max_{x\in[a,b]} f(x)$$

By Weierstrass theorem, $\exists x_1, x_2 \in [a,b]$ s.t.

$$m = f(x_1)$$
$$M = f(x_2)$$

Suppose $x_1 \le x_2$ Consider the interval $[x_1, x_2]$. We have

$$f(x_1) \le z \le f(x_2)$$

By applying the lemma to the interval $[x_1, x_2]$, $\exists c \in [x_1, x_2]$ s.t. $f(c) = z$

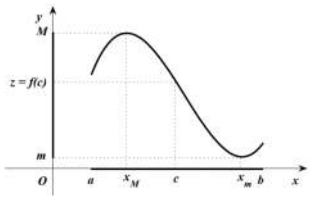
QED

Eg.

- The image of f is a compact interval:

$$\operatorname{Im} f = [m, M]$$

- Graphically:



Operators

lunedì 6 novembre 2023 16:33

$$f: A \subseteq R^n \to R^m$$

Example:

$f(x_1, x_2) = (x_1 + x_2, x_1 \cdot x_2)$

$f(1,2) = (3,2)$

Another way to see:

$f_1(x_1, x_2) = x_1 + x_2$

$f_2(x_1, x_2) = x_1 \cdot x_2$

$f = (f_1, f_2) = (x_1 + x_2, x_1 \cdot x_2)$

Another example:

$f: R^2 \to R^3$

$f(x_1, x_2) = (e^{x_1 + x_2}, x_1 \cdot x_2, x_1^2 + x_2^2)$

$f(1,2) = (e^3, 2, 5)$

Def.

An operator $f: A \subseteq R^n \to R^m$ is a $m -$ tuple $f(f_1, \dots, f_m)$ of functions of several variables

$f_i: A \subseteq R^n \to R, \forall i = 1,2, \dots, m$

defined by

$y_1 = f_1(x_1, \dots, x_n)$

$y_2 = f_2(x_1, \dots, x_n)$

...

$y_m = f_m(x_1, \dots, x_n)$

Definition

Let $f: A \subseteq R^n \to R^m$ be an operator and $x_0 \in R^n$ a limit point of A. We write

$\lim_{x \to x_0} f(x) = L \in R^m$

If, $\forall V_\varepsilon(L)\ \exists U_{\delta_\varepsilon}(x_0)$ of x_0 $s.t.$

$x_0 \neq x \in U_{\delta_\varepsilon}(x_0) \cap A \Rightarrow f(x) \in V_\varepsilon(L)$

The value L is called the limit of operator f at x_0

Proposition

An operator

$f = (f_1, \dots, f_m): A \subseteq R^n \to R^m$

is continuous at a point $x_0 \in A$ if all its component functions

$f_i \subseteq R^n \to R$

Are continuous

Algebra of operators:

- $f + g : \mathbb{R}^n \to \mathbb{R}^m$ is the operator defined by

$$(f+g)(x) = f(x) + g(x) \qquad \forall x \in \mathbb{R}^n$$

- αf is the operator defined by

$$(\alpha f)(x) = \alpha f(x) \qquad \forall x \in \mathbb{R}^n$$

Aggregate demand is the sum of agents' individual demands

- Aggregate demand is the sum of agents' individual demands

- Formally:

$$D = (D_1, \ldots, D_n) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$$

is given by

$$D = \sum_{i=1}^{m} D^i = D^1 + D^2 + \cdots + D^m$$

- This is a sum of operators

Linearity - I

lunedì 6 novembre 2023 17:27

Definition

A function $f: R^n \to R$ is said to be linear if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

$\forall x, y \in R^n$ and $\forall \alpha, \beta \in R$

Proposition

A function $f: R^n \to R$ is linear $\Leftrightarrow \begin{cases} f(x+y) = f(x) + f(y) \forall x, y \in R^n \\ f(\alpha x) = \alpha f(x), \forall x \in R^n, \forall \alpha \in R \end{cases}$

Eg.

$$f(x_1, x_2) = x_1 + x_2$$
$$x = (x_1, x_2), y = (y_1, y_2)$$
$$f(\alpha x + \beta y) = f\left(\alpha(x_1, x_2) + \beta(y_1, y_2)\right)$$
$$= f\left((\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)\right)$$
$$= f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$
$$= \alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2$$
$$= \alpha(x_1 + x_2) + \beta(y_1 + y_2)$$
$$= \alpha f(x) + \beta f(y)$$

Eg.

$$f(x) = \sum_{i=1}^{n} \chi_i x_i \, , \chi_i \in R$$

Proposition

$f: R^n \to R$ is linear $\Rightarrow f(0) = 0$

Proof

$f(0) = f(\alpha 0) = \alpha f(0), \forall \alpha \in R$

So

$f(0) = \alpha f(0), \forall \alpha \in R$

Hence

$f(0) = 0$

Property

$$f\left(\sum_{i=1}^{n} \alpha_i x^i\right) = \sum_{i=1}^{n} \alpha_i f(x^i)$$

$\forall \{x^i\}_{i=1}^{n}, \forall \{\alpha_i\}_{i=1}^{n}$

Definition

The set of all linear functions $f: R^n \to R$ is called the dual space of R^n and is denoted by $(\mathbb{R}^n)'$

Properties:

- Addition

  $\forall f, g \in (R^n)', (f+g)(x) = f(x) + g(x) \Rightarrow (f+g) \in (R^n)', \forall x \in R^n$
- Scalar multiplication

  $\forall f \in (R^n)', \alpha \in R^n: (\alpha f)(x) = \alpha f(x) \Rightarrow \alpha f \in (R^n)', \forall x \in R^n$

NB

The form of all linear function is $f(x) = \beta \cdot x, \forall x \in R$

Scalar product, hence in R^1

Riesz Theorem

$\forall x \in R^n$

A function $f: R^n \to R$ is linear $\Leftrightarrow \exists \chi \in R^n$ and is unique: $f(x) = \chi \cdot x$

Proof

Let $f: R^n \to R$ be linear

$f(e^i) = \chi_i$

Scalar product hence is in R^1

$(\Leftarrow)$

S.t. $f(x) = \chi x, \forall x \in R^n$

Set $\chi = \left(f(e^1), f(e^2), \ldots, f(e^n)\right)$

$$f(x) = f\left(\sum_{i=1}^{n} x_i e^i\right) = \sum_{i=1}^{n} x_i f(e^i) = \sum_{i=1}^{n} \chi_i x_i = \chi \cdot x, \forall x \in R^n$$

<u>Uniqueness:</u>

Assume $\exists \chi' \in R^n$. Hence , $\forall i = 1,2, \ldots, n$:

$\chi_i' = \chi' \cdot e^i = f(e^i) = \chi \cdot e^i = \chi_i$

$\Rightarrow \chi' = \chi,$ hence χ is unique

<u>Theorem:</u>

Linear functions are continuous

<u>Proof</u>

Take $f: R^n \to R$

Take $x^k \to x^0 \in R^n$

<u>Want to prove:</u> $f(x^k) \to f(x^0)$

By this

$\exists \chi \in R^n: f(x) = \chi \cdot x, \forall x \in R^n$

We have

$$|f(x^k) - f(x^0)| = \left|\sum_{i=1}^{n} \chi_i x_i^k - \sum_{i=1}^{n} \chi_i x_i^0\right| = \left|\sum_{i=1}^{n} \chi_i (x_i^k - x_i^0)\right| = \left|\chi \cdot \sum_{i=1}^{n} (x_i^k - x_i^0)\right|$$

$$= |\chi \cdot (x^k - x^0)| \leq ||\chi|| \cdot \left||x^k - x^0|\right|$$

Therefore

$$|f(x^k) - f(x^0)| \leq ||\chi|| \cdot \left||x^k - x^0|\right|$$

As $x^k \to x^0$, we have

$$||x^k - x^0|| \to 0$$

Hence

$$|f(x^k) - f(x^0)| \to 0$$

Therefore, we can declare:

$$x^k \to x^0 \Rightarrow f(x^k) \to f(x^0)$$

QED

Def:

Strictly increasing function:

$$x > y \Rightarrow f(x) > f(y)$$

Strongly increasing function:

$$x \gg y \Rightarrow f(x) > f(y)$$

Leontief function:

$f(x_1, x_2) = \min\{x_1, x_2\}$

Leontief in full generality:

$f(x_1, x_2, \dots, x_n) = \min\{x_1, x_2, \dots, x_n\}$

$x \gg y \Rightarrow f(x) > f(y)$

Proposition

Let $f: A \subseteq R^n \to R$

f strictly increasing $\Rightarrow f$ strongly increasing

Proof:

Let $x \gg y$

$$x \gg y \Rightarrow x > y \Rightarrow f(x) > f(y)$$

Hence, we have that

$$x \gg y \Rightarrow f(x) > f(y)$$

Definition:

Positivity

A function $f: A \subseteq R^n \to R$ is said to be

1. Positive if
$$x \geq 0 \Rightarrow f(x) \geq 0$$
2. Strictly positive if
$$x > 0 \Rightarrow f(x) > 0$$
3. Strongly positive if
$$x \gg 0 \Rightarrow f(x) > 0$$

Proposition:

$f: R^n \to R$, a linear function

f is (strictly, strongly) increasing $\Leftrightarrow f$ is (strictly, strongly) positive

Proof

$(\Rightarrow)$

Let f be increasing

To show: f is positive

Take $x \geq 0$

To show: $f(x) \geq 0$

$x \geq 0 \Rightarrow f(x) \geq f(0) = 0$

$(\Leftarrow)$

Let f be positive

To show: f is increasing

Take $x \geq y$

To show: $f(x) \geq f(y)$

Set $z = x - y$

$x \geq y \Rightarrow z \geq 0$

Since f is positive $\Rightarrow f(z) \geq 0$

By linearity, $f(z) = f(x) - f(y)$

$f(z) \geq 0 \Rightarrow f(x) - f(y) \geq 0 \Rightarrow f(x) \geq f(y)$

Hence, $x \geq y \Rightarrow f(x) \geq f(y)$

Riesz-Markov Theorem

$f: R^n \to R$ is lienar and increasing $\Leftrightarrow \exists \chi \in R^n_+$, unique, s.t. $f(x) = \chi \cdot x, \forall \in R^n$

In particular:

1. $\chi > 0 \Leftrightarrow f$ is strongly positive
2. $\chi \gg 0 \Leftrightarrow f$ is strictly increasing

Proof

$(\Leftarrow)$

Suppose $\chi \geq 0$.

WTS f increasing

By Riesz

$\mathrm{f}(x) = \chi \cdot x$

Take $x \geq y$,

WTS: $f(x) \geq f(y)$

Let $z = x - y$, as $\chi \geq 0$

$z \geq 0$ since $x \geq y$

Then $\chi \cdot z \geq 0$

Hence

$$f(x) - f(y) = \chi \cdot (x - y) = \chi \cdot z \geq 0 \Rightarrow f(x) \geq f(y)$$

$(\Rightarrow)$

Let f be increasing.

WTS: $\chi \geq 0$

Consider $e^i, e^i \geq 0 \Rightarrow f(e^i) \geq f(0) = 0$

Hence, $\forall i, \chi_i \geq 0 \Rightarrow \chi \geq 0$

Definition

An operator $T: R^n \to R^m$ is **linear** if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

$\forall x, y \in R^n, \forall \alpha, \beta \in R$

Properties

$T: R^n \to R^m$ is linear $\Leftrightarrow$

1. $T(x+y) = T(x) + T(y)$
2. $T(\alpha x) = \alpha T(x)$

$\forall x, y \in R^n, \forall \alpha, \beta \in R$

PROP If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $T(\mathbf{0}) = \mathbf{0}$

Denote by $L(R^n, R^m)$ the space of all linear operators

- When $m = 1$, so $R^m = R, L(R^n, R)$ reduces to $(R^n)'$

Theorem: Riesz generalized

An operator $T: R^n - R^m$ is linear $\Leftrightarrow \exists \chi$, a unique matrix $A_{m\times n}$ such that

$$T(x) = Ax$$

$\forall x \in R^n$

Proof

Relies on $x = \sum_{i=1}^{n} x_i \cdot e^i$

$A = \left(T_{m\times 1}(e^1), \dots, T_{m\times 1}(e^n)\right)$

Goes on like the smaller Riesz, though more cumbersome: non-examinable

Definition

Given two linear operators, $T: R^n \to R^m$ and $S: R^m \to R^q$, their product is the function $ST: R^n \to R^q$, defined by

$$(ST)(x) = S(T(x))$$

$\forall x \in R^n$

NB: If S and T are linear, so does the product ST.

Define $T: R^3 \to R^3$

$T(x) = (0, x_2, x_3)$

$A_{m\times n} = (T_{m\times 1}(e^1), T_{m\times 1}(e^2), T_{m\times 1}(e^3))$

$T(e^1) = T(1,0,0)$

$T(e^2) = T(0,1,0)$

$T(e^3) = T(0,0,1)$

$\Rightarrow A_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$T(x) = Ax$

Linearity - II

giovedì 9 novembre 2023 14:04

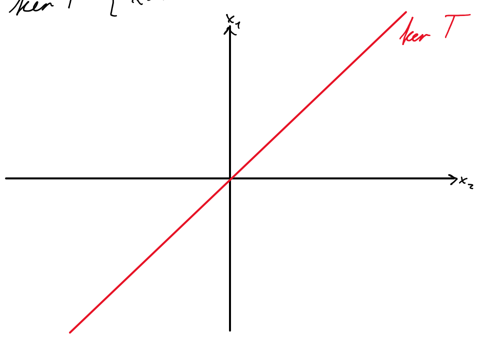
Def:
The **kernel**, denoted $\ker T$, of an operator $T: R^n \to R^m$ is the set

$$\ker T = \{x \in R^n : T(x) = 0\}$$

It is equivalent to say:

$$\ker T = T^{-1}(0)$$

Example:
$T: R^2 \to R$
$T(x_1, x_2) = x_1 - x_2$
$\ker T = \{x \in R^2 : T(x) = 0\} = \{x \in R^2 : x_1 = x_2\}$



Proposition
$T: R^n \to R^m$ is linear $\Rightarrow \ker T$ is a vector subspace of R^n

Proof:
Take $x, y \in \ker T$; $\alpha, \beta \in R$
WTS: $\alpha x + \beta y \in \ker T$
$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$
By hypothesis, $T(x) = T(y) = 0$
$\Rightarrow \alpha T(x) + \beta T(y) = \alpha \cdot 0 + \beta \cdot 0 = 0 \in \ker T$
We conclude that $\alpha x + \beta y \in \ker T$
Q.E.D.

Proposition
A linear $T: R^n \to R^m$ is injective $\Leftrightarrow \ker T$ is trivial, i.e., $\ker T = \{0\}$

Proof
$(\Rightarrow)$
Let T be injective
WTS: $\ker T = \{0\}$
Observe that $0 \in \ker T$ because $T(0) = 0$
Take $x \neq 0$
By injectivity,
$T(x) \neq T(0) = 0 \Rightarrow T(x) \neq 0, \forall x \neq 0$
Hence, 0 is the unique element of $\ker T$
$(\Leftarrow)$
Let $\ker T = \{0\}$
WTS: T is injective

Take $x \neq y$

WTS: $T(x) \neq T(y)$

We have

$x - y \neq 0 \Rightarrow x - y \notin \ker T$

Hence

$T(x - y) \neq 0$

By linearity

$T(x - y) = T(x) - T(y) \neq 0 \Rightarrow T(x) \neq T(y)$

Q.E.D.

Definition

The image (or range) or an operator $T: R^n \to R^m$ is

$\mathrm{Im}T = \{y \in R^m : y = T(x), \forall x \in R^n\}$

Definition

The rank

$\rho(T)$

Of a linear operator $T: R^n \to R^m$ is the dimension of $\mathrm{Im}T$

Definition

The nullity

$\nu(T)$

Of a linear operator $T: R^n \to R^m$ is the dimension of $\ker T$

Rank-Nullity theorem

Given a linear operator $T: R^n \to R^m$, we have

$\rho(T) + \nu(T) = n$

Corollary

$T: R^n \to R^m$ is a linear operator

T is injective $\Leftrightarrow T$ is surjective

Proof

$(\Rightarrow)$

Let T be injective, so $\ker T = \{0\}$

Hence $\nu(T) = 0$

By Rank-Nullity theorem, $n = \rho(T) + \nu(T) = \rho(T)$

So, $\rho(T) = n$

Hence, $\mathrm{Im}T = R^n$, so T is surjective

$(\Leftarrow)$

Let T be surjective

WTS: T is injective

T surjective $\Rightarrow \rho(T) = n$

By Rank-Nullity theorem

$n = \rho(T) + \nu(T) \Rightarrow \nu(T) = 0 \Rightarrow \ker T = \{0\}$

Hence T is injective

QED

Injective operator is also invertible

Invertible operator $T: R^n \to R^n$ has an inverse operator $T^{-1}: R^n \to R^n$

Proposition

An invertible linear operator $T: R^n \to R^n$ has a linear inverse $T^{-1}: R^n \to R^n$

Let A be the square matrix of T

The matrix of T^{-1} is the inverse matrix of A and is denoted A^{-1}

Consider a linear systems of equations, with n equations in n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

In matrix form:

$$A_{n\times n}x_{n\times 1} = b_{n\times 1}$$

Where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$x =$$

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$A \cdot x = b \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$T(x) = Ax$

$T(x) = b, b \in \text{Im}T \Leftrightarrow \exists x; (x) = b$

$T^{-1}(b) -$ preimage

If $T^{-1}(b)$ is a singleton $\Rightarrow$ unique solution

T surjective $\Leftrightarrow \forall b, \exists x \in T^{-1}(b)$

Cramer Theorem

Let A be a square matrix of order n. The linear system

$$A \cdot x = b$$

has one, and only one, solution $\forall b \in R^n$

$$\Leftrightarrow$$

A is invertible ($\det A \neq 0$)

In this case, the solution is given by $x = A^{-1}b$

Proof

($\Leftarrow$)

Let A be invertible. Then the operator $T(x) = Ax$ is invertible, hence surjective

A is surjective, so it is also injective.

($\Rightarrow$)

A is bijective, so it is surjective, so it is invertible.

Theorem

An operator $T: R^n \to R^m$ is linear $\Leftrightarrow \exists! A_{m\times n}$ such that $T(x) = A \cdot x, \forall x \in R^n$

Proof

($\Rightarrow$)

Assume that T is a linear operator from R^n to R^m $\left(\forall x, y \in R^n, \alpha, \beta \in R, T[\alpha x + \beta y] = \alpha T(x) + \beta T(y)\right)$

Set

$$A = \left[T[e^1], T[e^2], \dots, T[e^n]\right]$$

Where $B_{R^n} = \{e^1, e^2, \dots, e^n\}$ is the standard basis of R^n

Since $x \in R^n$ can be written as follows

$$x = \sum_{i=1}^{n} x_i e^i$$

We have

$$T(x) = T\left(\sum_{i=1}^{n} x_i e^i\right) = \sum_{i=1}^{n} x_i T(e^i)$$

$$= x_1 T(e^1) + x_2 T(e^2) + \dots + x_n T(e^n)$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \dots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \dots \\ a_{mn} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \dots + a_{1m} \cdot x_m \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \dots + a_{2m} \cdot x_m \\ \dots \\ a_{n1} \cdot x_1 + a_{n2} \cdot x_2 + \dots + a_{nm} \cdot x_m \end{bmatrix} = \begin{bmatrix} a^1 \cdot x \\ a^2 \cdot x \\ \dots \\ a^3 \cdot x \end{bmatrix} = A \cdot x$$

Therefore, A <u>exists</u>. Suppose the existence of another $B_{m\times n}$ such that $T(x) = B \cdot x, \forall x \in R^m$

$$\begin{cases} T(x) = A \cdot x \\ T(x) = B \cdot x \end{cases} \Rightarrow A \cdot x = B \cdot x \Rightarrow Ax - Bx = 0 \Rightarrow (A - B)x = 0 \Rightarrow A = B$$

($\Leftarrow$)

Let A be a matrix with dimension $m \times n$ such that $T(x) = Ax, \forall x \in R^m$

Let $x, y \in R^n$ and $\alpha, \beta \in R$, we have

$$T[\alpha x + \beta y] = A(\alpha x + \beta y) = A \cdot (\alpha x) + A \cdot (\beta y) = \alpha(Ax) + \beta(Ay) =$$
$$= \alpha T(x) + \beta T(y)$$

And T is a linear operator from R^n to R^m

Optimization

giovedì 9 novembre 2023 15:59

- maximum values are unique, whereas maximizers can be multiple
- Subset of the domain in which we are interested determines the existence of maximizers/maximum values/minimizers/minimal values

Definition

Let $f: A \subseteq R^n \to R$ be a real-valued function and C a subset of A. An element $\hat{x} \in C$ is called a **(global) maximizer** of f on C if

$$f(\hat{x}) \geq f(x), \forall x \in C$$

The value $f(\hat{x})$ of the function at $\hat{x}$ is called the **maximum value** of f on C.

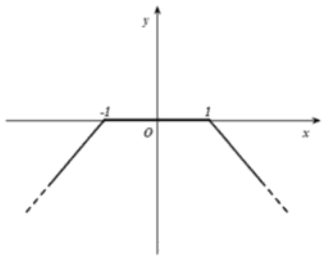
NB:
- Function f is the objective function
- Set C is the choice (feasible) set
- The maximum value of f is the maximum of the set
  $$\boldsymbol{f(C) = \{f(x): x \in C\} \subseteq R}$$
  i.e.
  $$\boldsymbol{f(\hat{x}) = \max f(C)}$$
- The maximum value is unique, denoted by
  $$\boldsymbol{\max_{x \in C} f(x)}$$

NB:
- Maximizers may not be unique
- The set is
  $$\boldsymbol{\left\{x \in C: f(x) = \max_{x \in C} f(x)\right\}}$$
- The set is denoted by
  $$\boldsymbol{\arg\max_{x \in C} f(x)}$$

Eg.

Define $f: R \to R$ by

$$f(x) = \begin{cases} -x + 1 \text{ if } x \geq 1 \\ 0 \text{ if } x \in (-1,1) \\ x + 1 \text{ if } x \leq -1 \end{cases}$$



If $C = R$

$$\max_{x \in R} f(x) = 0$$
$$\arg\max_{x \in R} f(x) = [-1, 1]$$

If $C = [1, +\infty)$

$$\max_{x \in [1,+\infty)} f(x) = 0$$
$$\arg\max_{x \in [1,+\infty)} f(x) = \{1\}$$

Notation of maximization problem

$$\max_x f(x) \text{ sub } x \in C$$

operation

constraint

„Subject to"

Properly written consumer problem:

$$\max_x u(x) \text{ sub } x \in B(p,w)$$

*w for wealth

Proposition

A function $\phi: A \subseteq R \to R$ is strictly increasing $\Leftrightarrow$

$s \geq t \Leftrightarrow \phi(s) \geq \phi(t), \forall s,t \in A$ $\quad (*)$

Proof

($\Leftarrow$)

Suppose $(*)$ holds

WTS: ϕ is strictly increasing

Let $s > t$

WTS: $\phi(s) > \phi(t)$

Suppose, *per contra*, that $\phi(s) < \phi(t)$. By $(*)$ we have $s \leq t$. But this contradicts the hypothesis $s > t$. We conclude that $\phi(s) > \phi(t)$

($\Rightarrow$)

Suppose ϕ is strictly increasing

WTS: (*) holds

Clearly, $s \geq t \Rightarrow \phi(s) \geq \phi(t)$

Now, take $\phi(s) \geq \phi(t)$

WTS: $s \geq t$

Suppose, per contra, $s < t$. Then $\phi(s) < \phi(t)$ since ϕ strictly increasing by hypothesis

But this contradicts $\phi(s) \geq \phi(t)$

So, (*) holds

Invariance theorem

Given a function $f: A \subseteq R^n \to R$, let $g: \text{Im} f \subseteq R \to R$ be a strictly increasing function.

The two problems of optimum

$$\max_{x} f(x) \text{ sub } x \in C \quad (**)$$

and

$$\max_{x} (g \circ f)(x) \text{ sub } x \in C$$

are equivalent, i.e., they have the same solutions

Proof

Suppose $\hat{x} \in \arg\max_{(x \in C)} f(x)$

WTS: $\arg\max_{(x \in C)} (g \circ f)(x)$

Since $\hat{x}$ solves the problem $(**)$

Then $f(\hat{x}) \geq f(x), \forall x \in C$

By the proposition,

$$f(\hat{x}) \geq f(x) \Leftrightarrow g(f(\hat{x})) \geq g(f(x))$$

Eg.

$u: R^2_{++} \to R: u(x_1, x_2) = x_1^a x_2^{1-a}$

$\max_x x_1^a x_2^{1-a}$ sub $x \in B(p, w)$

$g(t) = \lg t, t > 0$
$\tilde{u} = g \circ u = a \lg x_1 + (1-a) \lg x_2$
Log-linear utility function

$\max_x a \lg x_1 + (1-a) \lg x_2$ sub $x \in B(p, w)$

Theorem

Given $f: A \subseteq R^n \to R$, let C and C' be any two sets such that $C \subseteq C' \subseteq A$. We have

$$\max_{x \in C} f(x) \leq \max_{x \in C'} f(x)$$

Theorem

A function $f: A \subseteq R^n \to R$ continuous on a compact subset K of A admits (at least) a minimizer and (at least) a maximizer in K, i.e., $\exists x_1, x_2 \in K$ such that

$$f(x_1) = \max_{x \in K} f(x)$$

and

$$f(x_2) = \min_{x \in K} f(x)$$

Economic example:
$\max_x u(x)$ sub $x \in B(p, w)$
$u: R^n_+ \to R$
$\{x \in R^n_+ : p \cdot x \leq w\}$
The budget set is continuous and compact (closed and bounded).

$x_i^n \to +\infty \Rightarrow ||x^n|| \to +\infty$ – is possible only if the price of i is 0

Lemma

The budget set is compact if $\nexists$ free goods (i.e. $p \gg 0$)

Proposition

If $u: R^n_+ \to R$ is continuous and no free goods $(p \gg 0)$, then $\exists$ optimal bundles

Economic example

$u: R^2 \to R$ given by $u(x) = x_1 + x_2$
$p_1 > p_2 = 0$
$x = (x_1, x_2)$

$\max_x x_1 + x_2$ sub $p_1 x_1 + p_2 x_2 \leq w$

Considering that $p_2 = 0$

$\max_x x_1 + x_2$ sub $p_1 x_1 \leq w$

$\hat{x} = (\hat{x}_1, \hat{x}_2)$
$\hat{x} = (0, \hat{x}_2)$, but $\hat{x}_2 \to +\infty$, hence $\nexists$ optimal bundle

Concavity

martedì 14 novembre 2023 09:13

Convex sets
Take a collection of scalars:
$\alpha_1, \dots, \alpha_n$

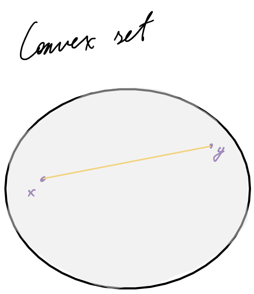
When are they weights?

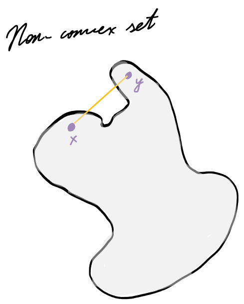
Definition: Weights
The elements of a collection of scalars $\{\alpha_i\}_{i=1}^{n}$ are called **weights** when
$$\alpha_i \geq 0$$
And
$$\sum_{i=1}^{n} \alpha_i = 1$$

Eg.
Suppose α_1, α_2 are weights
Hence,
$$\alpha_1, \alpha_2 \geq 0 \;\&\; \alpha_1 + \alpha_2 = 1$$
Hence, we can simplify as
$$\alpha, 1 - \alpha$$

Definition
A set C of R^n is convex if, $\forall x, y \in C, \forall \alpha \in [0,1]$
$$\alpha x + (1 - \alpha) y \in C$$

!!! Intervals are convex
!!! $B_\varepsilon(x_0), x_0 \in R^n$ is convex





Proposition
Budget sets are convex

Proof
Take $x, y \in B(p, w)$ and $\alpha \in [0,1]$
WTS: $\alpha x + (1 - \alpha) y \in B(p, w)$
$$p \cdot (\alpha x + (1 - \alpha) y) = \alpha \cdot p \cdot x + (1 - \alpha) \cdot p \cdot y \leq \alpha \cdot w + (1 - \alpha) w = w$$
So, $p \cdot (\alpha x + (1 - \alpha) y) \leq w$
So, $\alpha x + (1 - \alpha) y \in B(p, w)$

Definition
A linear combination
$$\sum_{i=1}^{n} \alpha_i x^i$$
is called a convex combination of the vectors $\{x^i\}_{i=1}^{n}$ if the coefficients α_i are weights

Eg
R^4
$$\alpha_1 = \frac{1}{8}$$
$$\alpha_2 = \frac{1}{8}$$
$$\alpha_3 = \frac{1}{4}$$
$$\alpha_4 = \frac{1}{2}$$

Definition
Given a finite collection of vectors $\{x^i\}_{i=1}^{k} \in R^n$
the **polytope** that they generate is the set

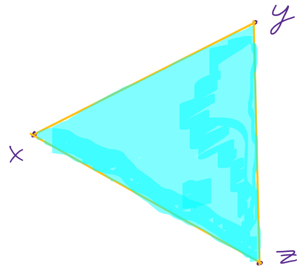
Definition

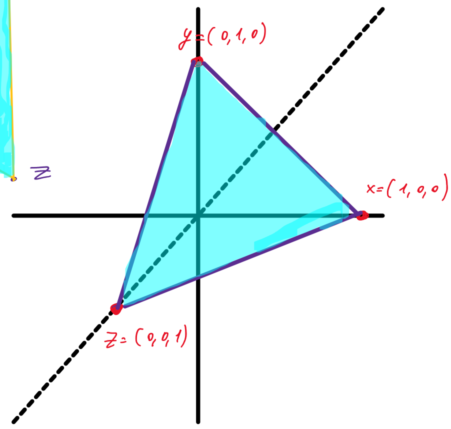
Given a finite collection of vectors $\{x^i\}_{i=1}^k \in R^n$

the **polytope** that they generate is the set

$$P = \left\{ \sum_{i=1}^{k} \alpha_i x^i : \sum_{i=1}^{k} \alpha_i = 1 \text{ and } \alpha_i \geq 0\ \forall i \right\}$$

Of all their convex combinations





Eg.

$$\{x^1, x^2, x^3\}$$

$$P = \left\{ \alpha_1 x^1 + \alpha_2 x^2 + \alpha_3 x^3 \mid \alpha_i \geq 0 \ \& \ \sum_{i=1}^{3} \alpha_i = 1 \right\}$$

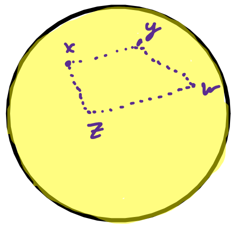
EG.

$$\{x, y, z\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$P = \{\alpha x + \beta y + (1 - \alpha - \beta) z : \alpha, \beta \geq 0, \alpha + \beta \leq 1\} =$$

$$= \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (1 - \alpha - \beta) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : \alpha, \beta \geq 0, \alpha + \beta \leq 1 \right\} =$$

$$= \left\{ \begin{pmatrix} \alpha \\ \beta \\ 1 - \alpha - \beta \end{pmatrix} : \alpha, \beta \geq 0, \alpha + \beta \leq 0 \right\}$$



Lemma

A set C in R^n is convex $\Leftrightarrow$ C is closed w.r.t. all convex combination of its own elements

In other words

The set is closed under all politops with the vertices as parts of the set

Proof

($\Leftarrow$)

Trivial using definition

($\Rightarrow$)

By induction:

Initial step: Let $n = 2$

It's true by the definition of convexity

Induction step

Suppose the result is true for $n - 1$

WTS: the result is true for n

Let $\{x^1, \dots, x^n\}$ be vectorsin C and $\{\alpha_1, \dots, \alpha_n\}$ weights

$$WTS: \sum_{i=1}^{n} \alpha_i x^i \in C$$

We have:

$$\sum_{i=1}^{n} \alpha_i x^i = \sum_{i=1}^{n-1} \alpha_i x^i + \alpha_n x^n = \frac{1 - \alpha_n}{1 - \alpha_n} \sum_{i=1}^{n-1} \alpha_i x^i + \alpha_n x^n =$$

$$= (1 - \alpha_n) \sum_{i=1}^{n-1} \frac{\alpha_i}{1 - \alpha_n} x^i + \alpha_n x^n =$$

$$= (1 - \alpha_n) \left(\frac{1}{(1 - \alpha_n)} \sum_{i=1}^{n-1} \alpha_i x^i \right) + \alpha_n x^n =$$

weights

$$= (1 - \alpha_n) \sum_{i=1}^{n-1} \alpha_i x^i + \alpha_n x^n$$

By induction hypothesis,

$$\sum_{i=1}^{n-1} \alpha_i x^i \in C$$

Hence, involving initial step,

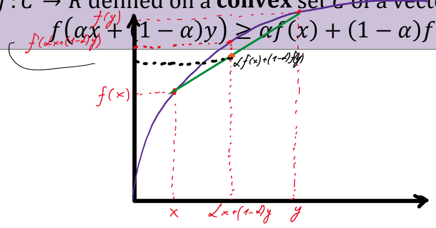
$$(1 - \alpha_n) \sum_{i=1}^{n-1} x^i + \alpha_n x^n \in C$$

QED

Def:

$f: C \to R$ defined on a **convex** set C of a vector space is *concave* if $\forall x, y \in C$, and $\alpha \in [0,1]$

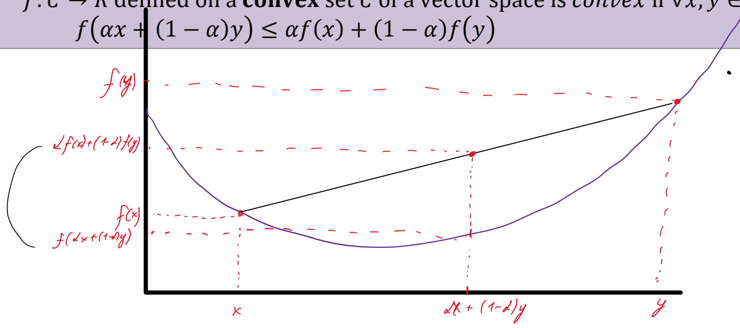
$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)$$



Def:

$f: C \to R$ defined on a **convex** set C of a vector space is *convex* if $\forall x, y \in C$, and $\alpha \in [0,1]$

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$



f is convex $\Leftrightarrow -f$ is concave

Def

A function $f: C \to R$ is strictly concave if $x, y \in C, x \neq y, \alpha = [0,1]$

$$f(\alpha x + (1-\alpha)y) > \alpha f(x) + (1-\alpha)f(y)$$

Def

A function $f: C \to R$ is strictly convex if $x, y \in C, x \neq y, \alpha = [0,1]$

$$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y)$$

f is strictly convex $\Leftrightarrow -f$ is strictly concave

Def

A function $f: C \to R$ is affine if it is both concave and convex, *i.e.*

$$f(\alpha x + (1-\alpha)y) = \alpha f(x) + (1-\alpha)f(y)$$

$\forall x, y \in C, \alpha = [0,1]$

Linear function:

$g(x) = mx$

Affine function:

$p(x) = mx + q = g(x) + q$

Proposition

A function $f: R^n \to R$ is affine $\Leftrightarrow \exists$ a linear function $l: R^n \to R$ and $\exists q \in R$ *s.t.*

$$f(x) = l(x) + q$$

$\forall x \in R^n$

NB:

Since $l(0) = 0$, we have $q = f(0)$

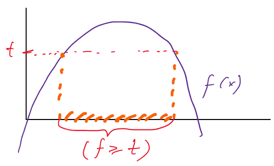
Def:

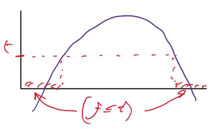
Superlevel or upper contour set

$f: A \subseteq R^n \to R$

Fix $t \in R$

$(f \geq t) = \{x \in A: f(x) \geq t\}$





Def:

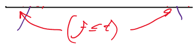
Sublevel or upper contour set:

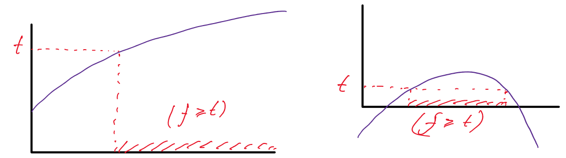
$f: A \subseteq R^n \to R$

Fix $t \in R$

$(f \leq t) = \{x \in A: f(x) \leq t\}$

Sublevel or upper contour set:
$f: A \subseteq R^n \to R$
Fix $t \in R$
$(f \leq t) = \{x \in A: f(x) \leq t\}$





Def:
Level set:
$f: A \subseteq R^n \to R$
Fix $t \in R$
$(f = t) = \{x \in A: f(x) = t\}$

Proposition
The superlevel sets of a concave function are convex

Proof
Let $f: C \to R$ be concave. Fix $t \in R$
WTS: $(f \geq t)$ is convex
Take $x, y \in (f \geq t)$ and $a \in [0,1]$
WTS: $ax + (1-a)y \in (f \geq t)$
We have
$$f(ax + (1-a)y) \geq af(x) + (1-a)f(y)$$
By hypothesis,
$$af(x) + (1-a)f(y) \geq at + (1-a)t = t$$
Hence,
$$f(ax + (1-a)y) \geq t$$
Therefore,
$$f(ax + (1-a)y) \in (f \geq t)$$
Q.E.D.

Jensen inequality
A function $f: C \to R$ defined on a convex set of R^n is concave $\Leftrightarrow$
$$f\left(\sum_{i=1}^{n} a_i x^i\right) \geq \sum_{i=1}^{n} a_i f(x^i)$$
$\forall \{x^i\}_{i=1}^{n} \in C$ and $\forall \{a\}_{i=1}^{n}$ – weights

Proof
$(\Leftarrow)$
Take $n = 2$ and you receive the definition of concavity
$(\Rightarrow)$
Let $f: C \to R$ be concave
$$\text{WTS: } f\left(\sum_{i=1}^{n} a_i x^i\right) \geq \sum_{i=1}^{n} a_i f(x^i)$$
We proceed by induction
Initial step: $n = 2$
Is true by the definition of concavity
Induction step:
Suppose true for $n - 1$:
$$f\left(\sum_{i=1}^{n-1} a_i x^i\right) \geq \sum_{i=1}^{n-1} a_i f(x^i)$$
WTS: true for n
$$f\left(\sum_{i=1}^{n} a_i x^i\right) = f\left(\sum_{i=1}^{n-1} a_i x^i + a_n x^n\right) = f\left(\frac{1-a_n}{1-a_n}\sum_{i=1}^{n-1} a_i x^i + a_n x^n\right) =$$
$$= f\left((1-a_n)\sum_{i=1}^{n-1} \underbrace{\frac{a_i}{1-a_n}}_{1} x^i + a_n x^n\right) = f\left((1-a_n)\underbrace{\sum_{i=1}^{n-1} \frac{a_i}{1-a_n} x^i}_{\in C} + a_n x^n\right) \geq$$
$$\geq (1-a_n) f\left(\sum_{i=1}^{n-1} \frac{a_i}{1-a_n} x^i\right) + a_n f(x^n) \geq (1-a_n)\left[\sum_{i=1}^{n-1} \frac{a_i}{1-a_n} f(x^i)\right] + a_n f(x^n) =$$
$$= \sum_{i=1}^{n-1} a_i f(x^i) + a_n f(x^n) = \sum_{i=1}^{n-1} a_i f(x^i)$$
Therefore

$$f\left(\sum_{i=1}^{n} a_i x^i\right) \geq \sum_{i=1}^{n} a_i f(x^i)$$

QED

Proposition

For a concave function $f: C \to R$ defined on a convex set of R^n, the set

$\arg\max_{x \in C} f(x)$

Is convex

Proof

$\arg\max_{x \in C} f(x) = \left(f \geq \max_{x \in C} f(x)\right)$

Superlevel sets are convex sets, hence $\arg\max_{x \in C} f(x)$ is also convex

Theorem

A strictly concave function $f: C \to R$ defined on a convex subset $C \subseteq R^n$ has at most 1 unique maximizer

Theorem

A strictly concave function $f: C \to R$ defined on a convex subset $C \subseteq R^n$ has at most a unique maximizer

Proof

Let f be strictly concave. Let $\hat{x}_1, \hat{x}_2 \in \arg\max_{x \in C} f(x)$

WTS: $\hat{x}_1 = \hat{x}_2$

Suppose, per contra, that instead $\hat{x}_1 \neq \hat{x}_2$. Consider:

$\frac{1}{2}\hat{x}_1 + \frac{1}{2}\hat{x}_2 \in C$

We have:

$$f\left(\frac{1}{2}\hat{x}_1 + \frac{1}{2}\hat{x}_2\right) > \frac{1}{2}\underbrace{f(\hat{x}_1)}_{=\max_{x \in C} f(x)} + \frac{1}{2}\underbrace{f(\hat{x}_2)}_{=\max_{x \in C} f(x)} = \frac{1}{2}\max_{x \in C} f(x) + \frac{1}{2}\max_{x \in C} f(x) = \max_{x \in C} f(x)$$

We have a contradiction:

$$f\left(\underbrace{\frac{1}{2}\hat{x}_1 + \frac{1}{2}\hat{x}_2}_{\in C}\right) > \max_{x \in C} f(x)$$

Therefore, the converse is true and $\hat{x}_1 = \hat{x}_2$ (f has at most one unique maximizer.)

Risk aversion

lunedì 20 novembre 2023 17:11

A list of possible outcomes: $\{c_1, \dots, c_n\}$
Each outcome c_i obtains a probability p_i
The following is the lottery:

$$\{c_1, p_1; \dots; c_n, p_n\}$$

$C = \{c_1, \dots, c_n\}$ set of all possible prizes
$\mathcal{L}$ − set of lotteries defined on C
We consider only monetary prizes, hence $C \subseteq R$

Expected utility criterion:

$$EU(L) = \sum_{i=1}^{n} u(c_i) p_i$$

Bernoullian utility function

$$u: C \to R$$

Notation

$\mathcal{L} \gtrsim \mathcal{L}'$

$\gtrsim$ −prefers or equal

$$L_1 = \left\{1, \frac{1}{2}; 5, \frac{1}{2}\right\}$$

$L_2 = \{5,1\}$
$EL_1 = EL_2 = 5$
$L_2 \gtrsim L_1$

Definition

DM (decision maker) is called:

1. ***Risk averse***, if, $\forall L \in \mathcal{L}$, it holds $E(L) \gtrsim L$
2. ***Risk loving***, if, $\forall L \in \mathcal{L}$, it holds $E(L) \lesssim L$
3. ***Risk neutral***, if, $\forall L \in \mathcal{L}$, it holds $E(L) \sim L$

Theorem

A DM with Bernoullian utility function u is risk averse $\Leftrightarrow$ u is concave

Proof

$(\Leftarrow)$

Suppose u is concave

WTS: DM is risk averse

We have:

$$u(EL) = u\left(\sum_{i=1}^{n} c_i p_i\right) \geq \sum_{i=1}^{n} p_i u(c_i) = EU(L)$$

Hence, $EL \gtrsim L$, so DM is RA

$(\Rightarrow)$

Same method, but vice-versa

Matrices - more

mercoledì 22 novembre 2023 14:49

NB

Let A e square matrix of order n

$[A|I_n] \sim \ldots \sim [I_n|A^{-1}]$

A is invertible with inverse matrix A^{-1}

Eg.

$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ Find, if exists, the inverse of A

$[A|I_n] = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} | \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - A$ is not invertible

1. Using the definition of inverse matrix: $A \cdot A^{-1} = A^{-1} \cdot A = I_n$
2. Using the elementary operations: $[A|I_n] \sim \ldots \sim [I_n|A^{-1}]$

Determinants

$A = [2], \det(A) = 2$

$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \cdot a_{22} + a_{12} \cdot a_{21}$

Properties of A,B square matrices with the same order n

a. If a line (row/column) of A is null, then $\det(A) = 0$

Eg: $\det \begin{bmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} = 0$

b. If B is found from A by multiplying a line by $k \in R$, then $\det(B) = k \cdot \det(A)$

Eg: $A = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}, \det(A) = 9$

$B = \begin{bmatrix} 4 & -2 \\ -1 & 5 \end{bmatrix}, \det(B) = 18$

c. If B is found from A by interchanging two parallel lines, then $\det(B) = -\det(A)$

Eg: $A = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}, \det(A) = 9$

$B = \begin{bmatrix} -1 & 5 \\ 2 & -1 \end{bmatrix}, \det(B) = -9$

d. If two parallel lines of A are equal, then $\det(A) = 0$

Eg:

$A = \begin{bmatrix} 2 & -1 & 3 \\ 2 & -1 & 3 \\ 3 & 3 & 1 \end{bmatrix}; \det(A) = 0$

e. If B is found from A by adding to a line(row/column) a multiple of a parallel line, then $\det(B) = \det(A)$

NB:

$\det[k \cdot A] = k^n \cdot \det(A); \det(A) = \det(A^T)$

Eg.

A square matrix of order 3 with $\det(A) \Rightarrow \det(2A) = 2^3 \det(A) = 8 \cdot 2 = 16$

Proposition: Let A be a square matrix of order n. Then:

A has a full rank $\Leftrightarrow \det(A) \neq 0$

NB: A is said to be a <u>singular matrix</u> if $\det(A) = 0$

Eg: $A = \begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}, \alpha \in R, \det(A) = \alpha^2 - 1$

A is singular $\Leftrightarrow \det(A) = 0 \Leftrightarrow \alpha^2 = \pm 1$

$\forall \alpha \in R \neq \{-1,1\}, \det(A) \neq 0, A$ – non-singular matrix

$\alpha \neq \pm 1, \det(A) \neq 0, A$ has a full rank, $\rho(A) = 2$

Corollary

Let A be a square matrix of order n. Then, the following conditions are equivalent

1. $S = \{x^1, x^2, \dots, x^n\}$ linearly independent $(x^j \in R^n, j = 1,2, \dots, n)$
2. $S = \{x^1, x^2, \dots, x^n\}$ linearly independent $(x^i \in R^n, i = 1,2, \dots, n)$
3. $\det(A) \neq 0$

Eg.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}, \det(A) = -6 \neq 0$$

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\} - \text{linearly independent}$$

$S = \{(2,1,3), (0,1,1), (0,2,-1)\}$ – linearly independent

Eg.

Is $S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ lin indept

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 3 & 1 \end{bmatrix}, \det(A) = -4 \neq 0$$

Theorem

Let A,B be square matrices with the same order n. Then

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

Eg.

$\det(A) = \frac{1}{2}; \det(B) = 2$

1. $\det(AB) = \det(A) \cdot \det(\mathrm{B}) = \frac{1}{2} \cdot 2 = 1$
2. $\det\left[\frac{1}{2} \cdot AB\right] = \left(\frac{1}{2}\right)^3 \det[AB] = \frac{1}{8} \cdot 1 = \frac{1}{8}$
3. $\det[A^2] = \det[A \cdot A] = \det(A) \cdot \det(A) = [\det(A)]^2$

   Hence,

   $\det(A^k) = [\det(A)]^k$

   $A^o = I_m$

Definition

Let A be a square matrix of order n. The real number a^*_{ij} $(i = 1, \dots, n; j = 1, \dots, n)$ given by

$$a^*_{ij} = (-1)^{i+j} \cdot \det(A_{ij})$$

Is called the **<u>algebraic complement/cofactor</u>** associated to a_{ij}

Eg.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 0 \\ 3 & 5 & 1 \end{bmatrix}$$

$$a_{21}^* = (-1)^{2+1} \cdot \det[A_{21}] = (-1)^{2+1} \cdot \det \begin{bmatrix} 1 & -1 \\ 5 & -1 \end{bmatrix} = -4 \neq 0$$

$A^* = \left[a_{ij}^*\right]$ – Matrix of algebraic complements or cofactor matrix associated with A

$[A^*]^T$ = Adjoint matrix associated to A

Eg.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$a_{11}^* = 1 \cdot \det(2) = 2$$
$$a_{12}^* = -1 \cdot \det(1) = -1$$
$$a_{21}^* = -1 \cdot \det(1) = -1$$
$$a_{22}^* = 1 \cdot \det(3) = 3$$

$$A^* = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}; [A^*]^T = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} - \text{Adjoint matrix}$$

Proposition:

Let A be a square matrix of order n. The determinant of A is equal to the sum of all the products of the elements of any line of A by their algebraic complements.

- $\forall i = 1, \dots, n: \det(A) = \sum_{j=1}^{n} a_{ij} a_{ij}^*$
- $\forall j = 1, \dots, n: \det(A) = \sum_{i=1}^{n} a_{ij} a_{ij}^*$

Eg.

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

$$\det(A) = 2 \cdot (-1)^{2+3} \det \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} = 2 \neq 0$$

A is not simple, $\rho(A) = 3$

Theorem (Laplace's Theorem)

Let A be a square matrix of order n. Then:

1. For any $i = 1, \dots, n, \sum_{j=1}^{n} a_{ij} \cdot a_{qj}^* = \begin{cases} \det(A) \text{ if } q = i \\ 0 \text{ if } q \neq i \end{cases}$
2. For any $j = 1, \dots, n, \sum_{j=1}^{n} a_{ij} \cdot a_{iq}^* = \begin{cases} \det(A) \text{ if } q = j \\ 0 \text{ if } q \neq j \end{cases}$

NB:

$$\begin{cases} \det(A^{-1}) = \dfrac{1}{\det(A)} \\ \det(A) \neq 0 \end{cases}$$

Theorem

A square matrix if invertible $\Leftrightarrow \det(A) \neq 0$

In this case, $A^{-1} = \frac{1}{\det(A)} \cdot [A^*]^T$

Square linear systems

Let A be a square matrix of order n. We consider the following matrix equation:

$$A \cdot x = b$$
$$x \in R^n, b \in R^n$$

$T: R^n \to R^n, T(x) = A \cdot x$ therefore $T(x) = b, b \in R^n$

- T is surjective: $Im(T) = R^n$
- T is not surjective: $Im(T) \subset R^n$

1. The linear system has a solution for a given $b \in R^n \Leftrightarrow b \in Im(T)$

   NB: The linear system has a solution $\forall b \in R^n \Leftrightarrow T$ is surjective

Eg.

$$\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\rho(A) = \rho(A|b)$$

$$\begin{cases} A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}, \rho(A) = 2 \\ A \neq 0, 1 \leq \rho(A) \leq 2 \end{cases}$$

$$\det(A) = -5 \neq 0$$

$$\begin{cases} [A|b] = \begin{bmatrix} 2 & 1 & b_1 \\ 3 & -1 & b_2 \end{bmatrix} \\ [A|b] \neq 0, \quad 1 \leq \rho(A) \leq 2 \\ \rho(A) \leq \rho(A|b) \end{cases}$$

$\rho(A) = \rho(A|b) = 2, \forall b \in R^2, b \in Im(T) =$ the linear system has at least one solution

2. The linear system has a unique solution for a given $b \in R^n \Leftrightarrow T^{-1}(b)$ is a singleton

   NB:

   The linear system has a unique solution $\forall b \in R^n \Leftrightarrow T$ is injective

Eg.

$$\begin{cases} 2x_1 + x_2 = 2 \\ x_1 - x_2 = 1 \end{cases}, [A|b] = \left[\begin{array}{cc|c} 2 & 1 & 2 \\ 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 1 \\ 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 1 \\ 0 & -\frac{3}{2} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0.5 & 1 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right], x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Eg.

Solve the linear system $\begin{cases} 2x_1 + x_2 = 2 \\ 4x_1 + 2x_2 = 1 \end{cases}$

$$\begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

1. $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

   $[A|b] = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$

   $A \neq 0, 1 \leq \rho(A) \leq 2; \det(A) = 0; \rho(A) = 1$

   $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}; \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}; \rho(A|b) = 2$

   $\rho(A) = 1 < 2 = \rho(A|b)$

   The linear system is impossible

2. $\begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

The linear system is impossible: $0x_1 + 0x_2 \neq -3$

Eg.

Solve $\begin{cases} 2x_1 + x_2 = 2 \\ 4x_1 + 2x_2 = 4 \end{cases}$ or $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

1. $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}, A \neq 0, 1 \leq \rho(A) \leq 2$

   $\det(A) = 0, \rho(A) = 1; [A|b\} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$

   $1 \leq \rho(A|b) \leq 2$

   $\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0; \begin{vmatrix} 2 & 2 \\ 4 & 4 \end{vmatrix} = 0; \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

   $\rho(A|b) = 1, \rho(A) = \rho(A|b) = 1$

   The linear system is possible

   $n = 2, \rho(A) = 1$

   $n - \rho(A) = 2 - 1 = 1 \rightarrow \infty$solutions with a free variable

   $$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \Rightarrow \begin{cases} 2x_1 + x_2 = 2 \\ 2x_1 = 2 - x_2 \Rightarrow x_1 = 1 - \frac{x_2}{2} \end{cases}$$

   $$\begin{cases} x_1 = 1 - \frac{t}{2} \\ x_2 = t \in R \end{cases}$$

2. $\begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.5 & 1 \\ 4 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Eg.

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ x_1 + x_2 - 2x_3 = 2 \end{cases} \text{ or } \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

1. $A \neq 0, 1 \leq \rho(A) \leq 2; \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0$

   $\rho(A) = 2; [A|b] = \left[\begin{matrix} 1 & -1 & 1 \\ 1 & 1 & -2 \end{matrix}\middle|\begin{matrix} 1 \\ 2 \end{matrix}\right], \rho(A) \leq \rho(A|b)$

   $1 \leq \rho(A|b) \leq 2, \rho(A|b) = 2$

   $\rho(A) = \rho(A|b) = 2$

   At least one solution exists (the linear system is possible)

2. $n = 3, \rho(A) = 2, n - \rho(A) = 3 - 2 = 1 \rightarrow \infty$ solutions

   $$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

   $$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} x_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow$$

   $$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \end{bmatrix} t \Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 - t \\ 2 + 2t \end{bmatrix}, t \in R \Rightarrow x = B^{-1} \cdot b^*$$

   $B(\exists B^{-1}), Bx = b^* \Rightarrow B^{-1}Bx = B^{-1}b^*$

   $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 - t \\ 2 + 2t \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 - t \\ 2 + 2t \end{bmatrix}$$

   ....

Theorem (Cramer)

Let A be a square matrix of order n. $Ax = b$ has a unique solution $\forall b \in R^n \Leftrightarrow A$ is invertible. In this case $x = A^{-1}b$

Matrices - continuation

mercoledì 22 novembre 2023 17:10

NB:

$M_{m\times n}(R)$ = set of $m \times n$ matries of real numbers
$M_{m\times n}(R)$ is a vector space

Definition

The rank of matrix A, $A \in M_{m\times n}(R)$ and we denote the set S as

$$S = \{x^j\}_{j=1}^{n}$$

With $x^j \in R^m$

The maximum subset of S linearly independent is called the rank of A: $\rho(A)$

Put another way,

Rank of a matrix A is the maximum number of its linearly independent columns

Eg.1

$$A = \begin{bmatrix} 3 & 6 & 18 & 2 \\ -1 & 2 & 6 & 4 \\ 0 & -1 & -3 & 6 \\ 2 & 1 & 3 & -8 \end{bmatrix}$$

Not all the columns are mutually independent as the third column is a multiple of the second column, hence $\rho(A) < 4$. Since all the other columns are mutually independent, we receive $\rho(A) = 3$

Eg. 2

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$S = \{x^j\}_{j=1}^{2}; x^j \in R^2$$

$x^1, x^2$are not collinear vectors, so S is linearly independent. Hence $\rho(A) = 2$

Example:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ \underbrace{2}_{x^1} & \underbrace{1}_{x^2} & \underbrace{3}_{x^3} \end{bmatrix}$$

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}, \rho(A) \geq 1$$

$$S_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \rho(A) \geq 2$$

$$S_3 = \left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{x^1}, \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{x^2}, \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{x^1+x^2} \right\}, \rho(A) = 2$$

Let $A \in M_{m\times n}(R)$ be the matrix associate to linear operator $T: R^n \to R^m$
Then $\rho(A) = \rho(T)$

Eg.

$T: R^2 \to R^2$

$$T(x) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_2 \end{bmatrix}$$
$$B_{R^2} = \{e^1, e^2\}$$
$$T(e^1) = T\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$T(e^2) = T\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$A = [T(e^1)|T(e^2)] = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$
$$T(x) = A \cdot x = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\rho(T) = \dim[Im(T)]$$
$$\dim[Im(T)] = 2 \Rightarrow \rho(T) = 2 \Rightarrow \rho(A) = 2$$

$\mathrm{Im}(T) = \{y \in R^2 : y = T(x), x \in R^2\}$

$Im(T) \subseteq R^2$(codomain of)

1. $\mathrm{Im}(T) = R^2 \Leftrightarrow T$ surjective
2. $\mathrm{Im}(T) \subset R^2 \leftrightarrow T$ isn't surjective

Corollary

A linear operator $T: R^n \to R^m$ with associated matrix $A \in M_{m\times n}$ is injective $\Leftrightarrow$ columns of A are independent

Example:

$T: R^3 \to R^2$

$$T(x) = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ x_2 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$A_{3\times 3} = [T(e^1), T(e^2), T(e^3)]$$
$$B_{R^3} = \{e^1, e^2, e^3\}$$

Theorem

For a matrix A, $\rho(A) = \rho(A^T)$

In other words:

For every matrix A, the maximum number of its linearly independent rows and columns coincide.

Eg.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, S = \left\{\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}, \rho(A) = 2$$
$$A^T = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}, S^* = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right\}, \rho(A^T) = 2$$

NB

For a linear operator $T: R^n \to R^m$ the following conditions are equivalent

1. T is injective
2. T is surjective
3. $\rho(A) = n$
4. $\rho(A^T) = n$

Properties:

- If $A_{m\times n}$ then $0 \le \rho(A) \le \min\{m, n\}$
- $A = 0 \Leftrightarrow \rho(A) = 0$
- If $A_{m\times n} \neq 0$ then $1 \le \rho(A) \le \min\{m, n\}$

Eg.

$$A_{3\times2} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}, A \neq 0$$

$$1 \leq \rho(A) \leq \min\{2,3\} = 2$$

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \right\} \text{ lin indep.} \Rightarrow \rho(A) = 2$$

Eg.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & -1 \\ 3 & 5 & 8 \end{bmatrix}, A \neq 0, 1 \leq \rho(A) \leq 3$$

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 8 \end{bmatrix} \right\}$$

$$S^* = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 8 \end{bmatrix} \right\}, 2 \leq \rho(A) \leq 3$$

S lin dep$\Rightarrow \rho(A) = 2$

T is not injective $\Leftrightarrow$ T is not surjective

Eg.

$T: R^2 \rightarrow R^2$

$$T(x) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A \neq 0, 1 \leq \rho(A) \leq 2$$

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}, \rho(A) \geq 1; S \text{ is lin dep}, \rho(A) = 1$$

$$A_x = 0$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{cases} \Rightarrow x_1 = -2x_2$$

$$\ker(T) = \left\{ \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix}, x_2 \in R \right\} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} t, t \in R \right\}$$

$$B_{\ker(T)} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \dim[\ker(T)] = 1 \right\}$$

Properties

$A, B \in M_{m\times n}(R)$

We have:

1. $\rho(A+B) \leq \rho(A) + \rho(B)$ and $\rho(\alpha A) = \rho(A), \forall \alpha \in R \neq 0$
2. $\rho(A) = \rho(CA) = \rho(AD) = \rho(CAD)$
   Where C, D are square matrices with full rank
3. $\rho(A) = \rho(A^T)$

Gaussian Elimination Procedure

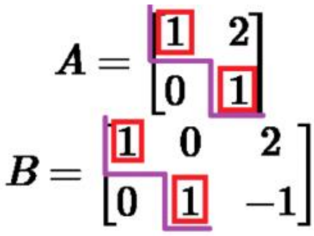
Echelon matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

We like these matrices because they immediately reveal their rank. They're called echolon matrices

Definition: echelon matrices:
1. The first element of each non-zero row is 1 (the pivot)
2. The entries of each column with pivot (1) are 0
3. The pivots form a staircase from the left to the right



Eg.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For any matrix $A_{m \times n}$ we can find its echelon form using elementary row operations
The elementary row operations we can perform are the following:
1. Multiplying any row by a non-zero scalar
2. Adding any row a multiple of any other row
3. Interchanging any two rows

eg.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & -1 \end{bmatrix}, A \neq 0, 1 \leq \rho(A) \leq 3$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 1 & 1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0.5 \\ 1 & 1 & 2 \\ 0 & 1 & -0.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 1.5 \\ 0 & 1 & -0.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 1.5 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 1.5 \\ 0 & 0 & 1 \end{bmatrix}$$

Let A be a square matrix of order n. The following conditions are equivalent:
1. A is invertible
2. $\rho(A) = n$ (A has full rank)
3. Two square matrices, B, C, exist such that $AB = CA = I_m$ and $B = C = A^{-1}$

Proposition

If the square matrices A and B of order n are invertible, then their product is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

NB:

$$[A|I_m] \sim \underbrace{\cdots}_{\text{Elementary operations}} \sim [I_m|A^{-1}]$$

Eg.

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, find A^{-1} using elementary row operations

$$\left[\begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}\middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right] \sim \left[\begin{matrix} 1 & 0.5 \\ 0 & 3 \end{matrix}\middle| \begin{matrix} 0.5 & 0 \\ -1 & 1 \end{matrix}\right] \sim \left[\begin{matrix} 1 & 0.5 \\ 0 & 1 \end{matrix}\middle| \begin{matrix} 0.5 & 0 \\ -1/3 & 2/3 \end{matrix}\right] \sim \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle| \begin{matrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{matrix}\right]$$

Hence, $A^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

Chapter 25: Forms

giovedì 23 novembre 2023 14:26

A function $f: R^n \to R$ given by

$$f(x_1, \dots, x_n) = K \cdot x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_n^{\alpha_n}$$

With $k \in R$ and $\alpha_i \in N$ is called a **monomial** of degree

$$m = \sum_{i=1}^{n} \alpha_i$$

Eg.

$$f: R^2 \to R, f(x_1, x_2) = x_1 \cdot x_2^3 (k = 1, \alpha_1 = 1, \alpha_2 = 3, m = 4)$$

$$g: R^2 \to R, g(x_1, x_2) = x_1 \cdot x_2^3 + x_2^4 \text{ ... FORM}$$

Definition

$f: R^n$
$\to R$ is a form if it is a sum of monomials with the same degree

NB.

A form $f: R^n \to R$ is said to be linear if

$$f(x) = \sum_{(i=1)}^{n} k_i x_i$$

Eg.

$$f: R^3 \to R, f(x) = 2x_1 - x_2 + x_3$$

Linear because we can write it as $f(x) = \chi x, \chi = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Checking for the properties:

Given $x, y \in R^3$

$$f(x+y) = 2(x_1 + y_1) - (x_2 + y_2) + (x_3 + y_3)$$
$$= 2x_1 - x_2 + x_3 + 2y_1 - y_2 + y_3 = f(x) + f(y)$$

Given $\alpha \in R, x \in R^3$

$$f(\alpha x) = 2(\alpha x_1) - (\alpha x_2) + (\alpha x_3) = \alpha(2x_1 - x_2 + x_3) = \alpha f(x)$$

NB:

$$f(0) = 0$$

Def:

A form $f: R^n \to R$ is said to be quadratic when it is the sum of monomials of second degree

Eg.

$f: R^2 \to R, f(x) = x_1^2 + 2x_1x_2 + 2x_2^2$ ... Quadratic form on R^2

Proposition

A bijective correspondence exists between quadratic forms symmetric matrices $A_{n \times n}$ such that

$$f(x) = x \cdot Ax = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j\,, \forall x \in R^n$$

Eg.

$f: R^2 \to R, f(x) = 2x_1^2 + 4x_1x_2 + x_2^2$

$$f(x) = x \cdot Ax = \underbrace{(x_1, x_2)}_{x} \cdot \left(\underbrace{\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} \right)$$

And A is called the symmetric matrix associated with the quadratic form

Eg.

$f: R^3 \to R$

$f(x) = x_1^2 + 4x_1x_2 + 2x_2^2 + x_3^2$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dots \text{Symmetric}$$

Def:

A quadratic form $f: R^n \to R$ is called:

a. Positive semi-definite if
   $\forall x \in R^n, f(x) \geq 0$

b. Negative semi-definite if
   $\forall x \in R^n, f(x) \leq 0$

c. Positive definite if
   $\forall x \in R^n - \{0\}, f(x) > 0$

d. Negative definite if
   $\forall x \in R^n - \{0\}, f(x) < 0$

e. Indefinite if
   $\exists x^1, x^2 \in R^n$ s.t. $f(x^1) < 0 < f(x^2)$

Eg.

a. $f: R^2 \to R, f(x_1, x_2) = x_1^2 + 2x_2^2$
   $\forall x \in R^2 - \{0\}, x_1^2 + 2x_2^2 > 0$, f is positive defined on R^2

b. $f: R^2 \to R, f(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2 = (x_1 + x_2)^2$
   $\forall x \in R^2, (x_1 + x_2)^2 \geq 0 \Rightarrow f$ is semi-definite positive

c. $f: R^2 \to R, f(x_1, x_2) = x_1^2 - x_1^2$:

$$x^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow f(x^1) = -1 < 0$$

$$x^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow f(x^2) = 1 > 0$$

   f is an indefinite quadratic form on R^2

A symmetric matrix A of order n is:

1. Positive (negative) semi-definite if $x \cdot Ax \geq 0$ $(x \cdot Ax \leq 0), \forall x \in R^n$
2. Positive (negative) definite if $x \cdot Ax > 0 (x \cdot Ax < 0), \forall x \in R^n - \{0\}$
3. Indefinite if two points $x^1, x^2 \in R^n$ exist such that $f(x^1) < 0 < f(x^2)$

NB

A is negative def. (semi def.) $\Leftrightarrow -A$ is positive def. (semi-def)

Eg.

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, x \cdot Ax = (x_1, x_2) \cdot \left(\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

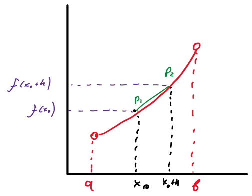
$= \left(x_1\ x_2\right) \cdot \left(2x_1 - x_2, -x_1 + 3x_2\right) = x_1\left(2x_1 - x_2\right) + x_2\left(-x_1 + 3x_2\right) =$
$= 2x_1^2 - x_1x_2 - x_1x_2 + 3x_2 = \cdots = x_1^2 + \left(x_1 - x_2\right)^2 + 2x_2^2 > 0, \forall x \in R^2 \neq 0$
A is positive def.

Proposition

A positive definite matrix is invertible. Its inverse A^{-1} is also positive definite $(x \cdot A^{-1}x > 0, \forall x \in R^n \neq 0)$

Chapter 27: Derivatives

giovedì 23 novembre 2023 15:24



Consider $f:(a,b) \to R$ and $x_0 \in (a,b)$.
We move from x_0 to $x_0 + h \in (a,b), h \neq 0$
Therefore f moves from $f(x_0)$ to $f(x_0+h)$.
The ratio given by

$$\frac{\Delta f}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0} = \frac{f(x_0+h) - f(x_0)}{h}$$

Is called the difference quotient of f w.r.t. the interval $[x_0, x_0+h]$ with $h \neq 0$

Geometric interpretation of $\frac{\Delta f}{\Delta x}$: It is the slope of the line passing through the points $(x_0, f(x_0))$ and $(x_0+h, f(x_0+h))$

Suppose that h becomes smaller and smaller $(h \to 0)$
Then $\frac{\Delta f}{\Delta x}$ can observe two different behaviors:

1. $\frac{\Delta f}{\Delta x} \to$ Real number
2. $\frac{\Delta f}{\Delta x} \to$ "Does not converge or diverges to infty"

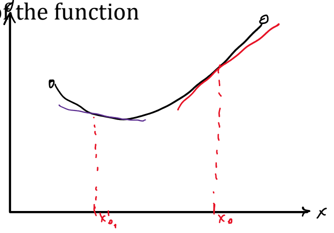
If $\frac{\Delta f}{\Delta x} \to l \in R$ as $h \to 0$ then f is said to be derivable at x_0 with derivative $f'(x_0) = l$

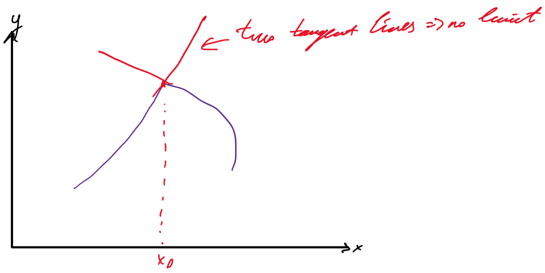
Definition
$f:(a,b) \to R$ is said to be derivable at $x_0 \in (a,b)$ if the limit

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Exists and is finite. The value of the limit $(f'(x_0))$ is called the derivative of f at x_0

Geo. Int.:
Suppose that f is derivable at x_0 with derivative $f'(x_0)$, then $f'(x_0)$ is the slope of the tangent line to the graph of the function





Eg.
$f: R \to R, f(x) = 2x^2$
Is f derivable at $x_0 \in R$?

$$\frac{\Delta f}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{h} = \frac{2(x_0+h)^2 - 2x_0^2}{h} = \frac{2(x_0^2 + x_0 h + h^2) - 2x^2}{h} = \frac{4x_0 h + 2h^2}{h}$$
$$= 4x_0 + 2h$$
$$\lim_{h \to 0} 4x_0 + 2h = 4x_0 \in R$$

Answer:

f is derivable at $x_0 \in R$ and $f'(x_0) = 4x_0$

NB

$x = x_0 + h$, hence, if $h \to 0$, then $x \to x_0$

f is derivable at x_0 if $\exists \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ and it is finite

Eg.

$f: R \to R, f(x) = 2x^2$

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} = \frac{2x^2 - 2x_0^2}{x - x_0} = \frac{2(x^2 - x_0^2)}{x - x_0} = 2(x + x_0) \to 2x_0$$

Eg.

$f: R \to R, f(x) = \sqrt[3]{x}$ Is f derivable at $x_0 = 0$

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} = \frac{\sqrt[3]{x} - \sqrt[3]{0}}{x - 0} = \frac{\sqrt[3]{x}}{x} = x^{-\frac{2}{3}}$$

$\nexists \lim\limits_{x \to 0} x^{-\frac{2}{3}}$, hence f is not derivable at $x_0 = 0$

Definition

$f: (a, b) \to R$ with domain of derivabililty $\Delta \subseteq (a, b)$.
Function $f': \Delta \to R$ such that any $x \in \Delta$ associates $f'(x) \in R$ is called the derivative function of f

Eg

$f: R \to R, f(x) = 2x^2, x_0 \in R$

$$\frac{\Delta f}{\Delta x} = 2(x + x_0) \xrightarrow[x \to x_0]{} 4x_0 \in R, \Delta = R$$

$f': R \to R, f'(x) = 4x$

Eg.

$f: R \to R, f(x) = \begin{cases} -x \text{ if x} < 0 \\ x \text{ if } x \geq 0 \end{cases} x_0 \in R$

- $x_0 \in (-\infty, 0): \dfrac{\Delta f}{\Delta x} = \dfrac{-x - 0}{x - 0} = -1 \xrightarrow[x \to x_0]{} -1$
- $x_0 \in (0, +\infty): \dfrac{\Delta f}{\Delta x} = \dfrac{x - x_0}{x - x_0} = 1 \xrightarrow[x \to x_0]{} 1$
- $x_0 = 0$: $\dfrac{\Delta f}{\Delta x} = \dfrac{f(x) - f(x_0)}{x - x_0} = \dfrac{f(x)}{x} = \begin{cases} -\dfrac{x}{x} \xrightarrow[x \to x_0]{} -1 \text{ if } x < 0 \\ \dfrac{x}{x} \xrightarrow[x \to x_0]{} 1 \text{ if } x > 0 \end{cases} \Rightarrow \nexists \lim\limits_{x \to 0} \dfrac{\Delta f}{\Delta x}$

  $\Rightarrow f$ is not derivable at $x_0 = 0$

$f': R - \{0\} \to R, f'(x) = \begin{cases} -1 \text{ if } x < 0 \\ 1 \text{ if } x > 0 \end{cases}$

One-sided derivatives

Definition

$f: (a, b) \to R$ is said to be derivable from the right side of $x_0 \in (a, b)$ if the one-sided limit

$$f'_+(x_0) = \lim_{n \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists and is finite. We say f is derivable from the right and the value of the value of the limit is $f'_+(x_0)$

Definition

$f: (a, b) \to R$ is said to be derivable from the left side of $x_0 \in (a, b)$ if the one-sided limit

$$f'_-(x_0) = \lim_{n \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists and is finite. We say f is derivable from the left and the value of the value of the limit is $f'_-(x_0)$

Eg.

$f: R \to R, f(x) = |x| = \begin{cases} -x, x < 0 \\ x, x > 0 \end{cases}, x_0 \in R$

- $f'_+(0) = \lim\limits_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^+} \dfrac{h-0}{h} = 1$
- $f'_-(0) = \lim\limits_{h \to 0^-} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^-} \dfrac{-h-0}{h} = -1$

NB

f is derivable at $x_0 \Leftrightarrow f'_+(x_0) = f'_-(x_0)$

Eg.

$f: R \to R, f(x) = \begin{cases} 0, x < 0 \\ x^2, x \geq 0 \end{cases}$

Is f derivable on R? Find the derivative function

- $x_0 \in (-\infty, 0), \lim\limits_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h} = \lim\limits_{h \to 0} \dfrac{0-0}{h} = 0$
  f is derivable at $x_0 \in (-\infty, 0)$ and $f'(x_0) = 0, \left(f'_+(x_0) = f'_-(x_0)\right)$
- $x_0 \in (0, +\infty), \lim\limits_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h} = \cdots = 2x_0$
  f is derivable at $x_0 \in (0, +\infty)$ and $f'(x_0) = 2x_0, \left(f'_+(x_0) = f'_-(x_0)\right)$
- At $x = 0$ we have
  $$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$
  1. $h > 0$: $\lim\limits_{h \to 0^+} \dfrac{h^2 - 0}{h} = \lim\limits_{h \to 0^+} \dfrac{h^2}{h} = \lim\limits_{h \to 0^+} h = 0 = f'_+(0)$
  2. $h < 0$: $\lim\limits_{h \to 0^-} \dfrac{0-0}{h} = 0 = f'_-(0)$

  $f'_+(0) = f'_-(0) = 0 \Rightarrow f$ is derivable at $x_0 = 0$ and $f'(0) = 0$

$$\Delta = R, f' = R \to R, f'(x) = \begin{cases} 0, x < 0 \\ 0, x = 0 \\ 2x, x > 0 \end{cases}$$

$$f'_+ R \to R, f'_+(x) = \begin{cases} 0, x < 0 \\ 2x, x \geq 0 \end{cases}, f'_- R \to R, f'_-(x) = \begin{cases} 0, x < 0 \\ 2x, x \geq 0 \end{cases}$$

NB:

If $f'(x_0)$ does not exist then x_0 is called a point of non-derivability of f

1. $f'_+ = L_1 \neq L_2 = f'_-(x_0)$ and G_f has a corner point at x_0
   $E.g.$
   $f(x) = |x|, f'_-(0) = -1 \neq 1 = f'_+(0)$
2. $\lim\limits_{(h \to 0)} \dfrac{f(x_0+h) - f(x_0)}{h} = \pm\infty$

NB

f is derivable on $(a, b) \Leftrightarrow f$ is derivable at any $x \in (a, b)$

Eg.

$f: [0, +\infty) \to R, f(x) = x^{\frac{1}{2}} = \sqrt{x}, A = [0, +\infty)$

- $x_0 \in (0, +\infty)$
  $\dfrac{\Delta f}{\Delta x} = \dfrac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \dfrac{1}{\sqrt{x} + \sqrt{x_0}} \xrightarrow[x \to x_0]{} \dfrac{1}{2\sqrt{x_0}} \in R$
  f is derivable at x_0 and $f'(x_0) = \dfrac{1}{2\sqrt{x}}$
- $x_0 = 0$ (Is f derivable at $x = 0$ from the right)
  $\dfrac{\Delta f}{\Delta x} = \dfrac{\sqrt{x} - \sqrt{0}}{x - 0} = \dfrac{\sqrt{x}}{x} = \dfrac{1}{\sqrt{x}} \xrightarrow[x \to 0]{} +\infty, f$ is not derivable at $x_0 = 0$

<u>Derivability and continuity</u>

Eg.

$f: R \to R, f(x) = |x|, f$ is continuous on $R, \lim\limits_{x \to x_0} |x| = |x_0| = f(x_0)$

f is not derivable at o

Proposition

$f:(a,b)\to R$ and $x\in(a,b)$. If f is derivable at $x\in(a,b)$ then f is continuous at $x\in(a,b)$

Proof

Assume that f is derivable at $x\in(a,b)$. We have to prove that f is continuous at x_0, i.e. $\lim_{x\to x_0} f(x)=f(x_0)$

$$\lim_{x\to x_0} f(x)=\lim_{x\to x_0}[f(x)-f(x_0)+f(x_0)]=\lim_{x\to x_0}[f(x)-f(x_0)]+\lim_{x\to x_0} f(x_0)=$$

$$=\lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}\cdot(x-x_0)\right]+f(x_0)=\lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}\right]\cdot\lim_{x\to x_0} x-x_0+f(x_0)$$

$$=$$

$$=f'(x_0)\lim_{x\to x_0}(x-x_0)+f(x_0)=f'(x_0)\cdot 0+f(x_0)=f(x_0)$$ and f is continuous on x_0

NB.

1. f is derivable at $x_0\in(a,b)\Rightarrow f$ is continuous at $x_0\in(a,b)$
2. f is not continuous at $x_0\in(a,b)\Rightarrow f$ is not derivable at $x_0\in(a,b)$
3. f is continuous at $x_0\in(a,b)\not\Rightarrow f$ is derivable at $x_0\in(a,b)$ (counterexample: $f(x)=\sqrt{x}$)

Derivatives of elementary functions

1. Constant functions

   $f:R\to R: f(x)=k\ (k\in R)$

   f is derivable at $x_0\in R$ and $f'(x_0)=0$:

   $$\frac{\Delta f}{\Delta x}=\frac{f(x)-f(x_0)}{x-x_0}=\frac{k-k}{x-x_0}=0$$

   $\Delta=R, f':\Delta\to R, f'(x_0)=0$

2. Power function $f:(0,+\infty)\to R, f(x)=x^a, a\in R$

   f is derivable at $x_0\in(0,+\infty)$ and $f'(x)=ax^{a-1}$

   Eg.

   $$f:[0,+\infty)\to R, f(x)=x^{\frac{1}{3}}$$

   $A=[0,+\infty)=\{0\}\cup(0,+\infty)$

   f is derivable on $(0,+\infty)$ and $f'(x)=\frac{1}{3}x^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{x^2}}$

   $$x_0=0:\frac{\Delta f}{\Delta x}=\frac{\sqrt[3]{x}-\sqrt[3]{0}}{x-0}=\frac{\sqrt[3]{x}}{x}=\frac{1}{\sqrt[3]{x^2}}\xrightarrow[x\to 0^+]{}+\infty$$

   f is not derivable at $x=0$

   $$f':(0,+\infty)\to R, f'(x)=\frac{1}{3\sqrt[3]{x^2}}$$

3. Exponential function: $f:R\to R, f(x)=a^x, a>0, a\neq 1$

   f is derivable on R and $f'(x)=a^x\cdot\ln a$

   $$\frac{\Delta f}{\Delta x}=\frac{a^x-a^{x_0}}{x-x_0}=a^{x_0}\cdot\frac{a^{x-x_0}-1}{x-x_0}\xrightarrow[x\to x_0]{}a^{x_0}\ln a$$

   Eg.

   $$f:R\to R, f(x)=\left(\frac{1}{2}\right)^x, f \text{ is derivable on } R$$

   $$f'(x)=\left(\frac{1}{2}\right)^x\cdot\ln\left(\frac{1}{2}\right)=2^{-x}\ln 2^{-1}=-\frac{\ln 2}{2^x}$$

   NB.

   $f:R\to R, f(x)=e^x, f$ is derivable on R and $f'(x)=e^x$

4. Elementary Trigonometric functions:

   $f:R\to R, f(x)=\sin x$, f is derivable on R and $f'(x)=\cos x$

   $f:R\to R, f(x)=\cos x$, f is derivable on R and $f'(x)=-\sin x$

5. Logarithmic function: $f:(0,+\infty)\to R, f(x)=\log_a x, (a>0, a\neq 1)$

   f is derivable on $(0,+\infty)$ and $f'(x)=\frac{1}{x\cdot\ln a}$

   $$\frac{\Delta f}{\Delta x}=\frac{f(x_0+h)-f(x_0)}{h}=\frac{\log_a(x_0+h)-\log_a x_0}{h}, x_0\in(0,+\infty), x_0+h\in(0,+\infty)$$

   $$=\frac{1}{h}\cdot\log_a\left(\frac{x_0+h}{x_0}\right)=\frac{1}{h}\log_a\left(1+\frac{h}{x_0}\right)=\frac{1}{x_0}\cdot\frac{x_0}{h}\log_a\left(1+\frac{h}{x_0}\right)=$$

   $$=\frac{1}{x_0}\cdot\left[\frac{\log_a\left(1+\frac{h}{x_0}\right)}{\frac{h}{x_0}}\right]\xrightarrow[h\to 0]{}\frac{1}{x_0}\cdot\frac{1}{\ln a}=\frac{1}{x_0\ln a}$$

   NB.

   $$\log_a e=\frac{\ln e}{\ln a}=\frac{1}{\ln a}$$

Algebra of derivatives

NB.:

$f,g:(a,b)\to R;\ \alpha,\beta\in R,\qquad \alpha f+\beta g:(a,b)\to R$

$(\alpha\in R,\beta=0):(\alpha f);(\alpha=\beta=1)(f+g)$

S

Proposition

$f,g:(a,b)\to R$ derivable at $x_0\in(a,b)$. Then $\alpha f+\beta g$ is derivable at $x_0\in(a,b),\forall\alpha,\beta\in R$ and

$$(\alpha f+\beta g)'(x_0)=\alpha f'(x)+\beta g'(x)$$

Proof

Assume that f,g are derivable at $x_0\in(a,b)$ and $\alpha,\beta\in R$. We have:

$$\frac{\Delta(\alpha f+\beta g)}{\Delta x}=\frac{(\alpha f+\beta g)(x)-(\alpha f+\beta g)(x_0)}{x-x_0}=$$

$$=\frac{\alpha\cdot f(x)+\beta\cdot g(x)-\alpha\cdot f(x_0)-\beta\cdot g(x_0)}{x-x_0}=$$

$$=\alpha\underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\xrightarrow[x\to x_0]{}f'(x_0)}+\beta\underbrace{\frac{g(x)-g(x_0)}{x-x_0}}_{\xrightarrow[x\to x_0]{}g'(x_0)}\xrightarrow[x\to x_0]{}\alpha f'(x_0)+\beta g'(x_0)\in R$$

$\alpha f+\beta g$ is derivable at $x_0\in(a,b)$ and $(\alpha f+\beta g)'(x_0)=\alpha f'(x_0)+\beta g'(x_0)$

Eg.

$f:R\to R, f(x)=x^2+\left(\frac{1}{4}\right)^x$ is derivable on R

$$f'(x)=2x+\left(\frac{1}{4}\right)^x\ln\frac{1}{4}=2x-\left(\frac{1}{4}\right)^x\ln 4$$

$$f'(0)=2*0-\left(\frac{1}{4}\right)^0\ln 4=-\ln 4$$

Proposition

$f,g:(a,b)\to R$ derivable at $x_0\in(a,b)$. The product $fg:(a,b)\to R$ is derivable at $x_0\in(a,b)$ and

$$(f\cdot g)'(x_0)=f'(x_0)\cdot g(x_0)+f(x_0)\cdot g'(x_0)$$

Proof

f,g derivable at $x_0\in(a,b)$. We have

$$\frac{\Delta(fg)}{\Delta x}=\frac{(fg)(x)-(fg)(x_0)}{x-x_0}=\frac{f(x)\cdot g(x)-f(x_0)\cdot g(x_0)}{x-x_0}=$$

$$=\frac{f(x)\cdot g(x)-f(x_0)\cdot g(x_0)+f(x_0)\cdot g(x)-f(x_0)\cdot g(x)}{x-x_0}=$$

$$=\frac{[f(x)-f(x_0)]g(x)+[g(x)-g(x_0)]f(x_0)}{x-x_0}=$$

$$=\underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\xrightarrow[x\to x_0]{}f'(x_0)}\underbrace{g(x)}_{\xrightarrow[x\to x_0]{}g(x_0)}+\underbrace{\frac{g(x)-g(x_0)}{x-x_0}}_{\xrightarrow[x\to x_0]{}g'(x_0)}\underbrace{f(x_0)}_{\xrightarrow[x\to x_0]{}f(x_0)}\xrightarrow[x\to x_0]{}f'(x_0)\cdot g(x_0)+g'(x_0)\cdot f(x_0)\in R$$

$\underbrace{fg\text{ is derivable at }x_0\in(a,b)}_{\text{Always check for derivability}}$ and $(fg)'(x_0)=f'(x_0)g(x)+f(x)g'(x_0)$

Eg.

$f:R\to R, f(x)=x^2e^x$, f is derivable

$f'(x)=2xe^x+x^2e^x=xe^x(2+x)$

$f'(x)=0\quad xe^x(2+x)=0\Rightarrow x(2+x)=0\Rightarrow x=0$ or $x=-2$

$\begin{cases}S=\{-2,0\}\text{ Set of stationary points of }f\\ x\notin S, x\text{ is not a stationary point of }f\end{cases}$

Proposition

$f,g:(a,b)\to R$ derivable at $x_0\in(a,b)$ and $g(x_0)\neq 0$

The ratio $\frac{f}{g}:(a,b)\to R$ is derivable at $x_0\in(a,b)$ and

$$\left(\frac{f}{g}\right)'(x_0)=\frac{f'(x_0)\cdot g(x_0)-f(x_0)\cdot g'(x_0)}{[g(x_0)]^2}$$

Eg.

$f:(0,+\infty)\to R, f(x)=\frac{\ln x}{x}$ f is continuous on $(0,+\infty)$ - because f is a product/ration of continuous functions on $(0,+\infty)$) " *and derivable on* $(0,+\infty)$" *for the same reason*"

$$f'(x)=\frac{\frac{1}{x}x-\ln x*1}{x^2}=\frac{1-\ln x}{x^2}$$
$$x+e^2, f'(e^2)=\frac{1-\ln e^2}{e^4}=-\frac{1}{e^4}<0$$
$$f'(x)=0, \frac{1-\ln x}{x^2}=0\Rightarrow 1-\ln x=0\Rightarrow x=e\Rightarrow S=\{e\}\text{ Unique statiary point}$$

NB

$$f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to R, f(x)=\tan x=\frac{\sin x}{\cos x}$$

$f(x)=\tan x$ is derivable on $x_0\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and

$$f(x)'=\frac{\cos x_0\cdot\cos x_0-\sin x_0\,[-\sin x_0]}{(\cos x_0)^2}=\frac{(\cos x_0)^2+(\sin x_0)^2}{(\cos x_0)^2}=\begin{cases}\dfrac{1}{(\cos x_0)^2}\\ 1+(\tan x_0)^2\end{cases}$$

Proposition

$f:(a,b)\to R$ and $g:(c,d)\to R$ with $f[(a,b)]\subseteq(c,d)$

If f is derivable as $x_0\in(a,b)$ and g is derivable at $f(x_0)\in(c,d)$ then $g\circ f$ is derivable at $x_0\in(a,b)$:

$$(g\circ f)'(x_0)=g'[f(x)]\cdot f'(x_0)$$

Proof

Assume that g is derivable at $y_0=f(x_0)$, we have

$$\lim_{k\to 0}\frac{g(x_0+k)-g(x_0)}{k}=g'(y)\Rightarrow\frac{g(x_0+k)-g(x_0)}{k}=g'(y)+o(1)\text{ as }k\to 0$$

Therefore we get $g(y_0+k)-g(y_0)=[g'(y)+o(1)]$ as $k\to 0$

Let h be a real number , $k\neq 0$. Set

$$k=f(x_0+h)-f(x_0)$$

Since f is derivable at x_0(f is also continuous), so

$$k\to 0\text{ as }h\to 0$$

We have:

$$g[f(x_0+h)]-g[f(x)]=[g'[f(x_0)]+o(1)]\cdot[f(x_0+h)-f(x_0)]\text{ as }h\to 0$$

It follows that

$$\frac{g[f(x_0+h)]-g[f(x_0)]}{h}=[g'[f(x_0)]+o(1)]\cdot\left(\frac{f(x_0+h)-f(x_0)}{h}\right)\text{ as }h\to 0$$

Taking the limit of each side for $h\to 0$

$$\lim_{h\to 0}\frac{g[f(x_0+h)]-g[f(x_0)]}{h}=g'[f(x_0)]\cdot f'(x_0)\in R$$

Theorem

$f:(0,+\infty)\to R, f(x)=x^\alpha,\alpha\in R, f$ is derivable on $(0,+\infty)$ and $f'(x)=\alpha x^{\alpha-1}$

Proof:

Let x be a strictly positive real number

We have:

$$f(x)=x^\alpha=e^{\ln x^e}=e^{\alpha\ln x}$$
$$g(x)=e^x, g'(x)=e^x$$
$$h(x)=\alpha\ln x, h'(x)=\frac{\alpha}{x}$$
$$f'(x)=g'[h(x)]\cdot h'(x)=g'[\alpha\ln x]\cdot\frac{\alpha}{x}=e^{\alpha\ln x}\cdot\frac{\alpha}{x}=e^{\ln x^\alpha}\cdot\frac{\alpha}{x}=x^\alpha\cdot\frac{\alpha}{x}$$
$$=\alpha\cdot x^{\alpha-1}$$

Theorem

$f:(a,b)\to R$ injective and derivable at $x_0\in(a,b)$. If $f'(x_0)\neq 0$, then f^{-1} is derivable at $y_0=f(x_0)$ and

$$(f^{-1})'(y_0)=\frac{1}{f'(x_0)}$$

Proof

Consider $f(x_0+h)=y_0+k$ and if $h\to 0$

$k\to 0$ (because function f is constant $\lim_{h\to 0}f(x_0+h)=f(x_0)=y_0$

From the definition of inverse function, $x_0=f^{-1}(y_0)$ and $x_0+h=f^{-1}(y_0+k)$

Therefore,

$$h=f^{-1}(y_0+k)-f^{-1}(y_0)$$

We have:

$$\frac{f(x_0+h)-f(x_0)}{h}=\frac{y_0+k-y_0}{f^{-1}(y_0+k)-f^{-1}(y_0)}=\frac{k}{f^{-1}(y_0+k)-f^{-1}(y_0)}=$$

$$= \frac{1}{\frac{f^{-1}(y_0+k) - f^{-1}(y_0)}{k}} \xrightarrow[k \to 0]{} \frac{1}{(f^{-1})(y_0)} \Rightarrow f'(x_0) = \frac{1}{(f^{-1})(y_0)} \Rightarrow$$

$$\Rightarrow (f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

Eg.

$f: R^2 \to R, f(x) = 2x_1 + x_2^2$ and $x^0 = (1,1) \in R^2$

Calculate, using the definition, the partial derivatives of f at x^0: $f'_{x_1}(x^0)$ and $f'_{x_2}(x^0)$

- $\lim_{h \to 0} \frac{f(x_1^0 + h, x_2^0) - f(x_1^0, x_2^0)}{h} = \lim_{h \to 0} \frac{[2(1+h) + 1^2] - [2 * 1 + 1^2]}{h}$
  $= \lim_{h \to 0} \frac{2 + 2h + 1 - 2 - 1}{h} = 2$
- f is partially derivable w.r.t. x_1 at x^0 and $f'_{x_1}(1,1) = 2$

$\lim_{h \to 0} \frac{f(x_1^0, x_2^0 + h) - f(x_1^0, x_2^0)}{h} = \lim_{h \to 0} \frac{[2 \cdot 1 + (1+h)^2] - [2 \cdot 1 + 1^2]}{h} = \lim_{h \to 0} \frac{2 + 1 + 2h + h^2 - 2 - 1}{h} = \lim_{h \to 0} \frac{h(2+h)}{h} = 2$

f is partially derivable w.r.t. x_2 at x^0 and $f'_{x_2}(1,1) = 2$

NB:

To calculate the partial derivatives of $f: R^n \to R; y = f(x_1, x_2, \dots, x_n)$ w.r.t. x_i $(i = 1,2, \dots, n)$ we apply all the derivation rules w.r.t. x_i keeping the other variable as constants

Eg.

$g: R^3 \to R$, $g(x) = (2x_1 - x_2^2)^3 + \ln(1 + x_3^2)$, $A = R^3$

- $g'_{x_1}(x) = 3(2x_1 - x_2^2)^2 \cdot 2 + 0 = 6(2x_1 - x_2^2)^2$
- $g'_{x_2}(x) = 3(2x_1 - x_2^2)^2(-2x_2) + 0 = -6x_2(2x_1 - x_2^2)^2$
- $g'_{x_3}(x) = 0 + \frac{1}{1 + x_3^2} \cdot 2x_3 = \frac{2x_3}{1 + x_3^2}$

$x^0 = (-1, 2, 0)$: $g'_{x_1}(x^0) = 6 \cdot (-6)^2 = 216$

$g'_{x_2}(x^0) = -12(-6)^2 = 216 \cdot (-2) = -432$

$g'_{x_3}(x^0) = 0$

$h: R^2 \to R$, $h(x) = x_1^2 \cdot e^{2x_1 + x_2^2}$ $A = R^2$

$h'_{x_1}(x) = 2x_1 \cdot e^{2x_1 + x_2^2} + x_1^2 \cdot e^{2x_1 + x_2^2} \cdot 2 = 2x_1 e^{2x_1 + x_2^2}[1 + x_1]$

$h'_{x_2}(x) = x_1^2 \cdot e^{2x_1 + x_2^2} \cdot (2x_2) = 2x_2 x_1^2 e^{2x_1 + x_2^2}$

Def:

Let $f: U \subseteq R^n \to R$ be a real function. The row vector of R^n $(\nabla f(x) \in R^n)$ given by

$$\nabla f(x) = [f'_{x_1}(x)\ f'_{x_2}(x) \dots f'_{x_n}(x)]$$

Is called the gradient of f at $x \in U$

Eg.

$f: R^3 \to R, f(x) == x_1^2 x_2 + x_2 * x_3 + 10, A = R^3$

$\nabla f(x) = [2x_1 x_2 \quad x_1^2 + x_3^2 \quad 2x_2 x_3], x \in R^3$

$x^0 = (-1, 1, -1) \in R^3$

$\nabla f(x^0) = [-2 \ \ 2 \ \ -2]$

NB

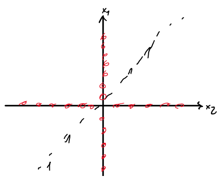
f is said to be partially derivable on $E \subseteq A$ when f is partially derivable at any point of $x \in E$

NB

f can be partially derivable at x^0 but at x^0 is not continuous

Eg.

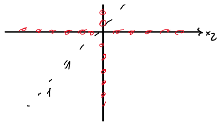
$f: R^2 \to R$, $f(x_1, x_2) = \begin{cases} 0 \text{ if } x_1 x_2 = 0 \\ 1 \text{ if } x_1 x_2 \neq 0 \end{cases}$ $x^0 = (0,0)$



$f(x_1, 0) = 0 \xrightarrow[(x_1, x_2) \to (0,0)]{} 0$

$f(0, x_2) = 0 \xrightarrow[(x_1, x_2) \to (0,0)]{} 0$

$f(x_1, x_2) = 1 \xrightarrow[(x_1, x_2) \to (0,0)]{} 1$



$f(0,x_2)=0 \xrightarrow[(x_1,x_2)\to(0,0)]{} 0$

$f(x_1,x_2)=1 \xrightarrow[(x_1,x_2)\to(0,0)]{} 1$

$\nexists \lim_{(x_1,x_2)\to(0,0)} f(x_1,x_2)$, f isn't continuous on $x^\circ=0$

Is f partially derivable at $x^\circ=(0,0)$

$f'_{x_1}(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$

$f'_{x_2}(0,0)=\lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$

$\nabla f(0,0)=[0 \quad 0]$, f is p-ly derivable, but f isn't continuous at $x^\circ=(0,0)$

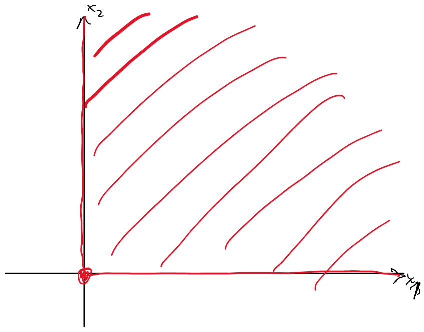
D

D = Subset of $A \subseteq R^n$ where $f: A \subseteq R^n \to R$ is partial derivable = domain of (partial) derivability of f

Def:

$f: U \subseteq R^n \to R$ with $D \subseteq U$ (open subset of R^n)

The operator $\nabla f: D \subseteq R^n \to R^m$ is called the derivative operator



Eg.

$f: R^2 \to R, , f(x) = \sqrt{x_1} + \sqrt{x_2}, A = \{x \in R^2: x_1 \geq 0 \wedge x_2 \geq 0\}$

$A = R_+^2 - \{0,0\}$

$\nexists f'(0)$

Not derivable at $\{0,0\}$

Eg.

$f: R^2 \to R, f(x) = x_1^2 x_2 + e^{x_1 x_2}$

$\nabla f: R^2 \to R^2, \nabla f(x) = [2x_1x_2 + e^{x_1x_2} \cdot x_2, x_1^2 + e^{x_1x_2}x_1]$

Definition

$f: U \subseteq R^n \to R$ (U open) is said to be differentiable at $x^0 \in U$ if a linear function $l: R^n \to R$ exists such that

$$f(x^0 + h) = f(x^0) + l(h) + o(||h||) \text{ as } ||h|| \to 0$$

$\forall x^0 + h \in U$

The linear function l is called the differential of f at x_0:

$$df(x^0): R^n \to R$$

Therefore, if f is differentiable at $x^0 \in U$ then

$$f(x^0 + h) = f(x^0) + df(x^0)(h) + o(||h||) \text{ as } h \to 0$$

Using the Riesz's theorem, the differentiable can be written as follows:

$df(x^0)(h) = \chi \cdot h$ with $\chi \in R^n$

That is the differential at x^0 is equal to the inner product of χ and h

NB:

$\chi = \nabla f(x^0)$

Theorem:

If $f: U \subseteq R^n \to R$ is differentiable at $x^0 \in U$ then f is **continuous** and **partially derivable** at x^0 and

$$df(x^0)(h) = \nabla f(x^0) \cdot h$$

$\forall h \in R^n$ and $x^0 + h \in U$

By substitution, we have:

$$f(x^0 + h) = f(x^0) + \nabla f(x^0) \cdot h + o(||h||) \text{ as } ||h|| \to 0$$

1. f is differentiable at $x^0 \Rightarrow f$ partially derivable at x^0
2. f differentiable at x^0 $\Rightarrow f$ is continuous at x^0 (f is not continuous at $x^0 \Rightarrow f$ is not differntiable at x^0)
3. f partially derivable at $x^0 \nRightarrow f$ continuous at x^0 $(n \geq 2)$

NB

$f: R^3 \to R, f(x) = x_1^3 + e^{x_2 + x_3^2}, f$ dif

$\nabla f: R^3 \to R^3, \nabla f(x) = \left[3x_1^2, e^{x_2+x_3^2}, e^{x_2+x_3^2} \cdot 2x_3\right]$

Theorem

Let f be a real function defined on $U \subseteq R^n$. If the partial derivatives of f are continuous then f is differentiable

Eg $f: R^2 \to R, f(x) = (x_1 + 2x_2)^2 \quad A = R^2$

Is f dif. on A?

$f'_{x_1}(x) = 2(x_1 + 2x_2)$ ... cont. on A

$f'_{x_2}(x) = 2(x_1 + 2x_2) \cdot 2$ ... cont on A

} f is dif. on $\forall x_0 \in A$

$x^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \nabla f(1,1) = [6, 12]$

$f(x^0 + h) = f(x^0) + \nabla f(1,1) \cdot h + o(\|h\|) \quad \|h\| \to 0$

$= 9 + (6 \ \ 12) \cdot (h_1 \ \ h_2) + o\left(\sqrt{h_1^2 + h_2^2}\right) \quad \|h\| \to 0$

$= 9 + 6h_1 + 12h_2 + o\left(\sqrt{h_1^2 + h_2^2}\right) \quad \|h\| \to 0$

$x^0 + h = x \Rightarrow h = x - x^0$

$f(x) = 9 + 6(x_1 - x_1^0) + 12(x_2 - x_2^0) + o(\|x - x_0\|)$ as $x \to x_0$

$= 9 + 6x_1 - 6 + 12x_2 - 12 + o(\|x - x_0\|)$

$= -9 + 6x_1 + 12x_2 + o(\|x - x_0\|)$

As $x \to x_0$, then $f(x) \approx 6x_1 + 12x_2 - 9$

NB:

total differential

$df(x^0) \cdot h = \nabla f(x^0) \cdot h = \left(f'_{x_1}(x^0), f'_{x_2}(x^0), \dots, f'_{x_n}(x^0)\right) \cdot (h_1 \ \ h_2 \ \dots h_n)$

$= f'_{x_1}(x^0)h_1 + f'_{x_2}(x^0)h_2 + \dots + f'_{x_n}(x^0) \cdot h_n$

$= f'_{x_1} \cdot (x_1 - x_1^0) + \dots + f'_{x_n} \cdot (x_n - x_n^0)$

partial dif-al of order 1   partial differential of order n

Partial derivatives of higher order

Eg

$f: R^2 \to R, f(x_1, x_2) = 2x_1^3 + x_2^2 + 10$

$f'_{x_1}(x) = 6x_1^2 \begin{cases} f''_{x_1x_1}(x) = 12x_1 \\ f''_{x_1x_2}(x) = 0 \end{cases}; f'_{x_2}(x) = 2x_2 \begin{cases} f''_{x_2x_1}(x) = 0 \\ f''_{x_2x_2}(x) = 2 \end{cases}$

Let f be a partial derivable function on $U \subseteq R^n$ (open subset of R^n)

$\begin{cases} \text{We know that } f'_{x_i}(x): U \to R \text{ with } i = 1,2,\dots,n \\ \qquad \text{First-partial derivative w.r.t. } x_i \end{cases}$

If f'_{x_i} is partially derivable at $x \in U$ then we can calculate the second-order partial derivative $w.r.t. x_j (j = 1,2,\dots,n)$

$f''_{x_ix_j}(x) (i, j = 1, \dots, n)$

Eg.

$g: R^3 \to R, g(x) = x_1^3 x_2 + 3x_3x_2 + 10$

$g'_{x_1}(x) = 3x_1^2 x_2$ →
- $g''_{x_1x_1}(x) = 6x_1x_2$
- $g''_{x_1x_2}(x) = 3x_1^2$
- $g''_{x_1x_3}(x) = 0$

$g'_{x_2}(x) = x_1^3 + 3x_3$ →
- $g''_{x_2x_1}(x) = 3x_1^2$
- $g''_{x_2x_2}(x) = 0$
- $g''_{x_2x_3}(x) = 3$

$g'_{x_3}(x) = 3x_2$ →
- $g''_{x_3x_1}(x) = 0$
- $g''_{x_3x_2}(x) = 3$
- $g''_{x_3x_3}(x) = 0$

$g'_{x_3}(x) = 3x_2$ → $g''_{x_3x_1}(x) = 0$, $g''_{x_3x_2}(x) = 3$, $g''_{x_3x_3}(x) = 0$

We can write all second-order partial derivatives in a square matrix

We can write all second-order partial derivatives in a square matrix

∇^2 $f(x)$

Which is called the Hessia matrix of f at $x \in U \subseteq R^n$

Eg: $\nabla g(x) = [3x_1^2 x_2 \quad x_1^3 + 3x_3 \quad 3x_2]$:

$$\nabla^2 g(x) = \begin{bmatrix} 6x_1x_2 & 3x_1^2 & 0 \\ 3x_1^2 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$$

$x^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\nabla g(x^0) = [3 \quad 4 \quad 3]$

$$\nabla^2 g(x^0) = \begin{bmatrix} 6 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$$

Theorem (Schwarz)

Let $f: U \to R$ be a function that has a second-order partial derivatives on U.

If the partial derivaties are continuous at $x \in U$ then

$$f''_{x_i x_j} = f''_{x_j x_i}$$

$\forall i, j = 1, \ldots, n$

Differentiability

martedì 28 novembre 2023 08:50

Differentiable functions

$f: (a, b) \to R$ and $x_0 \in (a, b)$. We want to approximate function f in a n-hood of x_0 using an affine function $r: R \to R$

The affine function which approximaes f at x_0:

$$f(x_0 + h) = r(x_0 + h) + o(h), h \to 0$$

With $x_0 + h \in (a, b)$

NB.

1. $r: R \to R, r(x) = mx + q \ (m, q \in R)$
2. $f(x_0) = r(x_0)(r(x_0) = mx_0 + q)$

We have

$$r(x_0 + h) = m(x_0 + h) + q = mx_0 + mh + q = r(x_0) + mh = f(x_0) + mh$$

$$f(x_0 + h) = f(x_0) + mh + o(h) \text{ as } h \to 0$$

And we indicate $l: R \to R, l(h) = mh$

Definition

$f: (a, b) \to R$. f is said to be **differentiable** at $x_0 \in (a, b)$ if a **linear function** $l: R \to R$ exists such that

$$f(x_0 + h) = f(x_0) + l(h) + o(h), h \to 0$$

$\forall x_0 + h \in (a, b)$

Eg.

$f: R \to R, f(x) = 2x^2$. Is f differentiable at x_0?

$$f(x_0 + h) = 2(x_0 + h)^2 = 2x_0^2 + \underbrace{4x_0}_{m} h + 2h^2 = f(x_0) + l(h) + o(h)$$

$$l: R \to R, l(h) = 4x_0 h$$

and it is differentiable at $x_0 \in R$ $(f$ is dif on $R)$

Definition

$l: R \to R$ is called the **<u>differential</u>** of f at x_0 and is indicated with $df(x_0)$ $(df(x_0): R \to R)$

We have:

$$f(x_0 + h) = f(x_0) + df(x_0)(h) + o(h), h \to 0$$

Theorem:

$f: (a, b) \to R$ is differentiable at $x_0 \in (a, b) \Leftrightarrow f$ is derivable at $x_0 \in (a, b)$.

The differential $df(x_0): R \to R$ is given by $df(x_0)(h)$

Proof

$(\Rightarrow)$

Assume that f is differentiable $f'(x_0) \cdot h$ at $x_0 \in (a, b)$, therefore by definition

$$f(x_0 + h) = f(x_0) + m \cdot h + o(h) \text{ as } h \to 0$$

We have

$$f(x_0 + h) - f(x_0) = mh + o(h) \text{ as } h \to 0$$

By multiplying each side by $\frac{1}{h}$

$$\frac{f(x_0 + h) - f(x_0)}{h} = m + \frac{o(h)}{h} \Rightarrow \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \left[m + \frac{o(h)}{h}\right]$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = m \in R$$

Hence, f is derivable at x_0 and $f'(x_0) = m$. The dif. of f at x_0 will be $df(x_0)(h) = m \cdot h = f'(x_0) \cdot h$

$(\Leftarrow)$

Assume that f is derivable at $x_0 \in (a,b)$, therefore

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = f'(x_0) \Rightarrow \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} - \underbrace{f'(x_0)}_{\lim_{h\to 0} f'(x_0)} = 0$$

o(h)

$$\Rightarrow \lim_{h\to 0}\left[\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0)\right] = 0$$

$$\Rightarrow \lim_{h\to 0}\left[\frac{f(x_0+h)-f(x_0)-f'(x_0)\cdot h}{h}\right] = 0$$

$$\Rightarrow f(x_0+h)-f(x_0)-f'(x_0)\cdot h = o(h), h\to 0$$

$$\Rightarrow f(x_0+h) = f(x_0)+f'(x_0)\cdot h + o(h) \text{ as } h\to 0$$

And f is differentiable at $x_0 \in (a,b)$ and $df(x_0)(h) = f'(x_0)\cdot h$

NB

$$\begin{cases} f: R\to R \\ f \text{ is dif at } x_0 \in (a,b) \text{ and } df(x_0)(h) = f'(x_0)h \end{cases}$$

Eg.

$f: R \to R, f(x) = x\cdot e^{2x}$ (because it is a product and composite of derivable functions at $x_0 \in R$)

Hence, f is differentiable at $x_0 \in R$

Proposition

$f:(a,b)\to R$. If f is differentiable at $x_0 \in (a,b)$ then f is continuous at $x_0 \in (a,b)$

Proof

Assume that f is differentiable at $x_0 \in (a,b)$, hence

$$f(x_0+h) = f(x_0)+f'(x_0)\cdot h + o(h) \text{ as } h\to 0$$

NB: *continuity is* $\lim_{x\to x_0} f(x) = f(x_0)$ or $\lim_{h\to 0} f(x_0+h) = f(x_0)$

We have

$$\lim_{h\to 0} f(x_0+h) = \lim_{h\to 0}[f(x_0)+f'(x_0)\cdot h + o(h)]$$

$$\lim_{h\to 0} f(x_0+h) = \lim_{h\to 0} f(x_0) + \lim_{h\to 0} f'(x_0)\cdot h + \lim_{h\to 0} o(h) = f(x_0)$$ and f is continuous at $x_0 \in (a,b)$

NB:

1. $$\begin{cases} f(x_0+h) = f(x_0)+f'(x_0)\cdot h + o(h), h\to 0 \\ x_0+h = x, h\to 0, x\to x_0 \\ f(x) = f(x_0)+f'(x_0)\cdot(x-x_0)+o(x-x_0) \end{cases}$$
2. $f:(a,b)\to R, D =$ domain of derivability of f

   $D \subseteq (a,b), f': D\to R$ and $E \subseteq D$

   $C^1(E)$ – set of all functions dif. (deriv) with continuity on E

EG.

$f: R\to R, f(x) = x^n, (n\in N), f \in C^1(R)$

$(f': R\to R$ and f' is cont on $R)$

$g: R\to R, f(x) = a^x\ (a>0, a\neq 1), g\in C^1(R)$

Higher order derivatives

$f:(a,b)\to R, f$ derivable on $D\subseteq (a,b)$ and $f': D\to R$ (Derivative function)

f' is said to be derivative at $x_0 \in D$ if

$$\lim_{h\to 0}\frac{f'(x_0+h)-f'(x_0)}{h} \text{ or } \lim_{x\to x_0}\frac{f'(x)-f'(x_0)}{x-x_0}$$

Exists and is finite. The value of the limit, indicated with $f''(x_0)$ is said to be the second derivative of f at x_0

Eg.

$f: R\to R, f(x) = 2x^2+4x, f\in C^1(R)$

$f'(x) = 4x+4$ Is f'derivable at $x_0 \in R$?

$$\lim_{h\to 0}\frac{f'(x_0+h)-f'(x_0)}{h}=\lim_{h\to 0}\frac{[4(x_0+h)+4]-[4x_0+4]}{h}=\lim_{h\to 0}\frac{4h}{h}=4\in R$$

f is twice derivable at $x_0 \in R$ and $f''(x_0)=4$

D' – domain of derivability of f' and $D' \subseteq D$

$f'': D' \to R$ Second derivative function

Eg.

$$f: R\to R, f(x)=\begin{cases} 0 \text{ if } x<0 \\ 2x^2 \text{ if } x\ge 0\end{cases}$$

f is derivable on $R-\{0\}$ with $f'(x)=0$ if $x<0$ and $f'(x)=4x$ if $x\ge 0$

If f derivable at $x_0=0$?

- $\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{0+0}{h}=0=f'_-(0)$
- $\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{(0+h)^2-0}{h}=\lim_{h\to 0^+}\frac{2h^2}{h}=0=f'_+(0)$

$$f': R\to R, f'(x)=\begin{cases} 0, x<0 \\ 0, x=0 \\ 4x, x>0\end{cases}$$

f' is derivable on $R-\{0\}$ with $f''(x)=0$ if $x<0$ and $f''(x)=4$ if $x>0$

Is f' derivable at $x=0$?

$$\lim_{h\to 0^-}\frac{f'(0+h)-f'(0)}{h}=\lim_{h\to 0^-}\frac{0-0}{h}=0=f''_-(0)$$

$$\lim_{h\to 0^+}\frac{f'(0+h)-f'(0)}{h}=\lim_{h\to 0^+}\frac{4(0+h)-0}{h}=4=f''_+(0)$$

$f''_-(0)=0\ne 4=f''_+(0)\Rightarrow f'$ is not derivable at $x=0$ $(\nexists f''(0))$

$$f'': R-\{0\}\to R, f''(x)=\begin{cases} 0 \text{ if } x<0 \\ 4 \text{ if } x>0\end{cases}$$

Eg.

$f: R\to R, f(x)=x^2e^{2x}; f\in C^2(R)$

$f'(x)=2x\cdot e^{2x}+x^2\cdot e^{2x}\cdot 2=e^{2x}(2x+2x^2)$

$f''(x)=e^{2x}(2x+2x^2)+e^{2x}(2+4x)=e^{2x}[4x+4x^2+2+4x]=e^{2x}(4x^2+8x+2)$

Definition

$f:(a,b)\to R.\ n-1$ times differentiable at $x_0\in R$

f is said to be differentiable n times at x_0 if the limit

$$\lim_{h\to 0}\frac{f^{n-1}(x_0+h)-f^{n-1}(x_0)}{h}$$

Exists and is finite. The value of the limit, indicated with $f^{(n)}(x_0)$ is said to be the derivative of f of order n at x_0

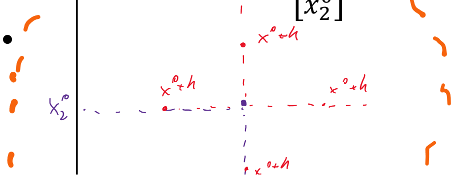
NB:

1. $n=1, f^{(0)}=f$
2. $C^n(E)=$ Set of all the functions differentialbe n times on E with continuity

Partial derivatives

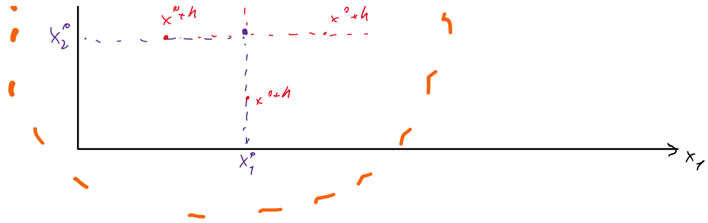
$f: U\subseteq R^n\to R, U$ is an open subset of R^n

- $f: U\subseteq R^2\to R, x^0=\begin{bmatrix}x_1^0\\x_2^0\end{bmatrix}$
-



$$x^0+he^1=\begin{bmatrix}x_1^0\\x_2^0\end{bmatrix}+\begin{bmatrix}h\\0\end{bmatrix}=\begin{bmatrix}x_0^1+h\\x_2^0\end{bmatrix}$$

$$x^0+he^2=\begin{bmatrix}x_1^0\\x_2^0\end{bmatrix}+\begin{bmatrix}0\\h\end{bmatrix}=\begin{bmatrix}x_0^1\\x_2^0+h\end{bmatrix}$$



$$= \begin{bmatrix} x_2^0 \end{bmatrix} + \begin{bmatrix} h \end{bmatrix} = \begin{bmatrix} x_2^0 + h \end{bmatrix}$$

Definition

$f: U \subseteq R^n \to R$ and $x^0 \in U$. f is said to be partially derivable at $x^0 \in U$ if

$$\forall i = 1,2,\dots,n, \lim_{h\to 0} \frac{f\left(x^0 + he^i\right) - f(x^0)}{h}$$

Exists and are finite. The values of these limits are called the partial derivatives of f at x_0

NB:

$f'_{x_1}\left(x^0\right)$ or

$\dfrac{\delta f}{\delta x_1}(x^0)$

Chapter 28 - Differential methods

giovedì 30 novembre 2023 13:47

NB

x^0 is said to be a local maximizer of f on A if $\exists B_\varepsilon(x^0): x \in B_\varepsilon(x^0) \cap A, f(x) \le f(x^0)$
x^0 is said to be a local minimizer of f on A if $\exists B_\varepsilon(x^0): x \in B_\varepsilon(x^0) \cap A, f(x) \ge f(x^0)$

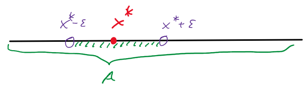
Theorem (Fermat's theorem)

$f: A \subseteq R \to R$ and $x^* \in A$ **is an interior point**
If f is differentiable at $x^* \in A$ and x^* is either local maximizer or local minimizer of f on A then $f'(x^*) = 0$

Proof

For a local minimizer

Assume that $x^* \in A$ is a local minimizer of f on A $\left(x^* \text{ is an interior point of } A\right)$ and f is differentiable at $x^* \in A$ $\left(\exists f'(x^*)\right)$



From the assumption, $\exists B_\varepsilon(x^*)\ s.t. \forall x \in B_\varepsilon(x^*) \cap A, f(x^*) \le f(x)$
Taking $h \in (0,\varepsilon)\ s.t. x^* + h \in B_\varepsilon(x^*)$ we have

$$\begin{cases} \dfrac{f(x^*+h)-f(x^*)}{h} \ge 0 \\ \forall h \in (0,\varepsilon) \end{cases} \Rightarrow \lim_{h\to 0^+} \frac{f(x^*+h)-f(x^*)}{h} \ge \lim_{h\to 0^+} 0 \Rightarrow \lim_{h\to 0^+} \frac{f(x^*+h)-f(x^*)}{h} \ge 0$$

$\Rightarrow f'_+(x^*) \ge 0$

Taking $h \in (-\varepsilon, 0) s.t. x^* + h \in B_\varepsilon(x^*)$ we have

$$\begin{cases} \dfrac{f(x^*+h)-f(x^*)}{h} \le 0 \\ \forall h \in (-\varepsilon,0) \end{cases} \Rightarrow \lim_{h\to 0^-} \frac{f(x^*+h)-f(x^*)}{h} \le \lim_{h\to 0^-} 0 \Rightarrow \lim_{h\to 0^-} \frac{f(x^*+h)-f(x^*)}{h} \le 0$$

$\Rightarrow f'_-(x^*) \le 0$

Since f is dif. at x^* we have $f'_-(x^*) = f'_+(x^*) = f'(x^*) = 0$

Same strategy for a local maximizer

Assume that $x^* \in A$ is a local maximizer of f on A $\left(x^* \text{ is an interior point of } A\right)$ and f is differentiable at $x^* \in A$ $\left(\exists f'(x^*)\right)$
From the assumption, $\exists B_\varepsilon(x^*)\ s.t. \forall x \in B_\varepsilon(x^*) \cap A, f(x^*) \ge f(x)$
Taking $h \in (0,\varepsilon)\ s.t. x^* + h \in B_\varepsilon(x^*)$ we have

$$\begin{cases} \dfrac{f(x^*+h)-f(x^*)}{h} \le 0 \\ \forall h \in (0,\varepsilon) \end{cases} \Rightarrow \lim_{h\to 0^+} \frac{f(x^*+h)-f(x^*)}{h} \le \lim_{h\to 0^+} 0 \Rightarrow \lim_{h\to 0^+} \frac{f(x^*+h)-f(x^*)}{h} \le 0$$

$\Rightarrow f'_+(x^*) \le 0$

Taking $h \in (-\varepsilon, 0) s.t. x^* + h \in B_\varepsilon(x^*)$ we have

$$\begin{cases} \dfrac{f(x^*+h)-f(x^*)}{h} \ge 0 \\ \forall h \in (-\varepsilon,0) \end{cases} \Rightarrow \lim_{h\to 0^-} \frac{f(x^*+h)-f(x^*)}{h} \ge \lim_{h\to 0^-} 0 \Rightarrow \lim_{h\to 0^-} \frac{f(x^*+h)-f(x^*)}{h} \ge 0$$

$\Rightarrow f'_-(x^*) \ge 0$

Since f is dif. at x^* we have $f'_-(x^*) = f'_+(x^*) = f'(x^*) = 0$

NB:

$f: A \subseteq R \to R, x_0$ interior point of A

-
- $\begin{cases} f \text{ dif at } x_0 \in A \\ x_0 \text{ is a local max or local min} \end{cases} \Rightarrow f'(x^*) = 0$
- $f'(x^*) \ne 0 \Rightarrow x^*$ is not a local max/min or f is not dif at x^*
- f dif at x^* and $f'(x^*) \ne 0 \Rightarrow x^*$ is not a local maximizer/minimizer for f

The solution of the equation $f'(x) = 0$, if exist, are called stationary points of f

Eg.

54

$f: (0, +\infty) \to \mathbb{R};\quad f(x) = \frac{\ln x}{x} \quad f \in C^1((0,+\infty))$

$f'(x) = \dfrac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \dfrac{1-\ln x}{x^2}, \quad f'(x) = 0$

$$\begin{cases} \frac{1-\ln x}{x^2} = 0 \Rightarrow 1-\ln x = 0 \Rightarrow \ln x = 1 \Rightarrow x = e \\ \forall x > 0 \end{cases}$$

$S = \{e\}$; $e^2 \in A$, $f'(e^2) \neq 0$ and e^2 isn't stationary $\Rightarrow$ e^2 isn't max/min on f

---

Eg.

$f: R \to R$, $f(x) = (2x^2-1)^2$, $f \in C^1(R)$

$f'(x) = 2(2x^2-1)\cdot(4x) = 8x(2x^2-1)$

$f'(x)=0 \mid 8x(2x^2-1)=0 \Rightarrow 16x^3 = 8x \Rightarrow \left[\begin{array}{l} x=0 \\ 2x^2=1 \end{array}\right. \Rightarrow \left[\begin{array}{l} x=0 \\ x=\pm\sqrt{\frac{1}{2}} \end{array}\right.$

$S = \{-\sqrt{\frac{1}{2}}, 0, \sqrt{\frac{1}{2}}\}$

$\forall x \in R$, $x \notin S$, x isn't a local max/min of f on R

We can generalize Fermat's theorem:

Theorem (Fermat generalized on R^n)
$f: A \subseteq R^n \to R$ and $x^* \in A$ interior point of A
If f is differentiable at x^* and x^* is a local mazimizer or minimizer of f on A then $\nabla f(x^*) = 0$

NB:
$f: A \subseteq R^n \to R, x^*$ internal point of A

f dif at x^* and x^* is a local max/min of f on A $\Rightarrow \nabla f(x^*)$

Eg

$f: R^3 \to R$, $f(x) = x_1^2 + 3x_2^2 + (x_3-1)^2$

f is dif. on R^3

$\nabla f(x) = [\,2x_1 \quad 6x_2 \quad 2(x_3-1)\,]$

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\nabla f(x) = 0$ Stat. condition

$$\begin{cases} 2x_1 = 0 \\ 6x_2 = 0 \\ 2(x_3-1)=0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 1 \end{cases} \Rightarrow S = \{(0,0,1)\},$$

$\hat{x} = (-1,0,1)$, $\nabla f(\hat{x}) = [-2 \;\; 0 \;\; 0] \neq 0$

Theorem (Rolle's theorem)
Let f be a real function continuous on $[a,b]$ differentiable on (a,b) and $f(a) = f(b)$
At least a point $\exists x^* \in (a,b)$ s.t. $f'(x^*) = 0$

Proof
Assume that f is continuous on $[a,b]$ differentiable on (a,b) and $f(a) = f(b)$
From Weiestrass theorem, two points $x_1, x_2 \in [a,b]$ exist s.t.

$$\begin{cases} m = f(x_1) = \min_{x\in[a,b]} f(x) \\ M = f(x_2) = \max_{x\in[a,b]} f(x) \end{cases}$$

1. $m = M$: the function is constant on $[a,b] \Rightarrow f'(x) = 0, \forall x \in (a,b)$
2. $m < M$ and M is an interior point $\Rightarrow f$ is not constant on $[a,b]$
   Suppose that $x_2 \in (a,b) \Rightarrow f'(x_2) = 0 \Rightarrow x_2 \in S$
3. $m < M$ and m is an interior point $\Rightarrow f$is not constant on $[a,b]$
   Suppose that $x_1 \in (a,b) \Rightarrow f'(x_1) = 0 \Rightarrow x_1 \in S$

Eg: (1) $f: [-1,1] \to R$, $f(x) = x(x^2-1)$

f cont on $R \Rightarrow f$ cont on $[-1,1]$

f dif on $R \Rightarrow f$ dif on $(-1,1)$

$f(-1) = (-1)[(-1)^2-1] = 0 = 1\cdot(1^2-1) = f(1)$

f dif on $\mathbb{R} \Rightarrow f$ dif on $(-1,1)$

$f(-1) = (-1)[(-1)^2 - 1] = 0 = 1\cdot(1^2-1) = f(1)$

we can apply Rolle's theorem to f on $[-1,1]$, $\exists\, x^* \in (-1,1)$ s.t. $f'(x^*) = 0$

(2) find $x^* \in (a,b)$ s.t. $f'(x^*) = 0$

$f'(x) = 1\cdot(x^2-1) + x\cdot 2x = x^2 - 1 + 2x^2 = 3x^2 - 1$

$f'(x) = 0$: $3x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x_{1,2} = \pm\sqrt{\frac{1}{3}}$

NB

Can we extend the Rolle's theorem in an open interval $[a,+\infty)$, $(-\infty,b]$ or $(-\infty,+\infty)$

f cont on $[a,+\infty)$

f dif on $(a,+\infty)$

$f(a) = \lim\limits_{x\to+\infty} f(x)$

Limits should be equal

Theorem (Mean Value theorem)

Let f be a real function continuous on $[a,b]$ and diff. on (a,b). At least a point $x^* \in (a,b)$ exists such that

$$f'(x^*) = \frac{f(b)-f(a)}{b-a} \text{ or } f(b)-f(a) = f'(x^*)\cdot(b-a)$$

Proof

Let g be a real function defined on $[a,b]$ given by

$$g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]$$

That is g is the difference between f and the affine function such that its graph passing through points $(a, f(a))$ and $(b, f(b))$

Function g is continuous on $[a,b]$ (dif of cont. func. on $[a,b]$) and diff on (a,b)(diff between differ functions on (a,b))

We have:

$$g(a) = f(a) - \left[f(a) + \underbrace{\frac{f(b)-f(a)}{b-a}(a-a)}_{=0}\right] = 0$$

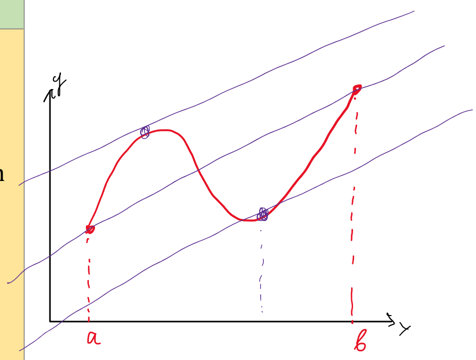
$$g(b) = f(b) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(b-a)\right] = f(b) - f(a) - f(b) + f(a) = 0$$

Therefore $g(a) = g(b) \Rightarrow$ we can apply Rolle's theorem on $[a,b]$: at least a point x^* exists such that $g'(x^*) = 0$

The derivative of g is

$$\begin{cases} g'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a} \\ g'(x^*) = 0 \end{cases} \Rightarrow \begin{cases} g'(x^*) = f'(x^*) - \dfrac{f(b)-f(a)}{b-a} \\ g'(x^*) = 0 \end{cases} \Rightarrow$$

$$\Rightarrow 0 = f'(x^*) - \frac{f(b)-f(a)}{b-a} \Rightarrow f'(x^*) = \frac{f(b)-f(a)}{b-a}$$



NB:

Geometric interpretation of $f'(x^*) = \frac{f(b)-f(a)}{b-a}$

1. $f'(x^*)$ Slope of the tangent line to the G_f at x^*
2. $\dfrac{f(b)-f(a)}{b-a}$ Slope of the line passing through $(a, f(a))$ and $(b, f(b))$

Eg $f : [-1, \ln 4] \to \mathbb{R}$; $f(x) = e^x$ $\quad f \in C^1(\mathbb{R})$

f is cont. and dif on $[-1, \ln 4]$ and $f'(x) = e^x$

$\exists\, x^* \in (-1, \ln 4)$: $f'(x^*) = \dfrac{f(\ln 4) - f(-1)}{\ln 4 - (-1)}$

$e^{x^*} = \dfrac{e^{\ln 4} - e^{-1}}{\ln 4 + 1} \Rightarrow x^* = \ln\left[\dfrac{4 - e^{-1}}{\ln 4 + 1}\right]$

Eg $f:[0,16]\to R$, $f(x)=\sqrt{x}$

I. f is cont on $[0,16]$, dif on $(0,16)$

We can apply the mean value theorem to $f:[0,16]$

at least a point $x^*\in(0,16)$ s.t. $f'(x^*)=\frac{f(16)-f(0)}{16-0}$

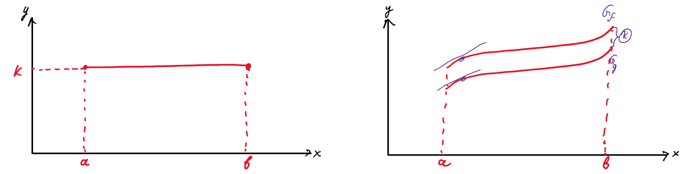
II. find x^*

$f(x)=\sqrt{x} \Rightarrow f'(x)=\frac{1}{2\sqrt{x}}$

$\frac{1}{2\sqrt{x}}=\frac{\sqrt{16}-\sqrt{0}}{16-0} \Rightarrow \frac{1}{2\sqrt{x}}=\frac{1}{4} \Rightarrow \sqrt{x}=2 \Rightarrow x^*=4$

From the Mean Value Theorem we have two important corollaries.

(a)



Corollary

Let f be a real function cont. on $[a,b]$ and diff on (a,b).
Then

$$\forall x\in(a,b), f'(x)=0 \Leftrightarrow \exists k\in R: \forall x\in[a,b], f(x)=k$$

Corollary

Let f,g be real functions, continuous on $[a,b]$ and differentiable. Then

$$\forall x\in(a,b), f'(x)=g'(x) \Leftrightarrow \exists k\in R: \forall x\in[a,b], f(x)=g(x)+k$$

Eg Let f be a diff. f on R with $f(x)\neq 0$ s.t. $\forall x\in R$, $f'(x)+4x^3[f(x)]^2=0$ and $f(0)=1$

Find $f(x)$:

$\forall x\in R$, $f'(x)+4x^3[f(x)]^2=0 \Rightarrow f'(x)=-4x^3[f(x)]^2$

$\Rightarrow -\frac{f'(x)}{[f(x)]^2}=4x^3 \Rightarrow \left[\frac{1}{f(x)}\right]'=[x^4]'$ $\forall x\in R$

$\Rightarrow \frac{1}{f(x)}=x^4+K$, $K\in R$

At $x=0$, we have $\frac{1}{f(0)}=0^4+K \Rightarrow \frac{1}{1}=0+K \Rightarrow K=1$

By substitution, we have $\frac{1}{f(x)}=x^4+1 \Rightarrow f(x)=\frac{1}{x^4+1}$

Theorem: generalization of Mean Value Theorem

Let f be a real function. If f is $n-1$ times continuously differentiable on $[a,b]$ and n times differentiable on (a,b) then $\exists x^*\in(a,b)$ s.t.

$$f(b)-f(a)=\sum_{k=1}^{n-1}\frac{f^k(a)}{k!}(b-a)^k+\frac{f^{(n)}(x^*)}{n!}(b-a)^n$$

$n=1: f(b)-f(a)=f(x^*)(b-a)$

$n=2: f(b)-f(a)=f'(a)(b-a)+\frac{1}{2}f''(x^*)\cdot(b-a)^2$

Monotonicity and differentiability

NB:

f is increasing on $(a,b), \forall x_1, x_2 \in (a,b)$

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$$

f is strictly increasing on $(a,b), \forall x_1, x_2 \in (a,b)$

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

f is decreasing on $(a,b), \forall x_1, x_2 \in (a,b)$

$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$$

f is strictly decreasing on $(a,b), \forall x_1, x_2 \in (a,b)$

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

Proposition:

f differentiable on (a,b) with $a,b \in \bar{R}$

f is increasing on $(a,b) \Leftrightarrow f'(x) \ge 0, \forall x \in (a,b)$

Proof

$(\Rightarrow)$

Assume that f is increasing on (a,b) and let x be an interior point of (a,b)

$\forall h > 0$, we have $f(x+h) \ge f(x)$. Therefore we have

$$\frac{\overbrace{f(x+h)-f(x)}^{positive}}{\underbrace{h}_{positive}} \ge 0 \Rightarrow \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \ge \lim_{h\to 0^+} 0 \Rightarrow$$

$$\Rightarrow \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \ge 0 \Rightarrow f'(x) \ge 0$$

$\big(f'_+(x) = f'_-(x) = f'(x)\big)$ therefore $\forall x \in (a,b), f'(x) \ge 0$

$(\Leftarrow)$

Assume $f'(x) \ge 0, \forall x \in (a,b), a,b \in \bar{R}$

Let x_1, x_2 be two different points of (a,b) s.t. $x_1 < x_2$

NB: f dif on $(a,b) \Rightarrow f$ dif on $[x_1,x_2] \Rightarrow f$ is continuous $\Rightarrow$ We can apply the minimum value theorem to f on $[x_1,x_2], \exists x^* \in (x_1,x_2)$ s.t.

$$f'(x^*) = \frac{f(x_2)-f(x_1)}{\underbrace{x_2-x_1}_{>0}} \ge 0 \Rightarrow f(x_2)-f(x_1) \ge 0 \Rightarrow f(x_2) \ge f(x_1)$$

And f is increasing on (a,b)

Proposition:

f differentiable on (a,b) with $a,b \in \bar{R}$

f is decreasing on $(a,b) \Leftrightarrow f'(x) \le 0, \forall x \in (a,b)$

Proof

$(\Rightarrow)$

Assume that f is decreasing on (a,b) and let x be an interior point of (a,b)

$\forall h > 0$, we have $f(x+h) \le f(x)$. Therefore we have

$$\frac{\overbrace{f(x+h)-f(x)}^{negative}}{\underbrace{h}_{positive}} \le 0 \Rightarrow \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \le \lim_{h\to 0^+} 0 \Rightarrow$$

$$\Rightarrow \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \le 0 \Rightarrow f'(x) \le 0$$

$\big(f'_+(x) = f'_-(x) = f'(x)\big)$ therefore $\forall x \in (a,b), f'(x) \le 0$

$(\Leftarrow)$

Assume $f'(x) \le 0, \forall x \in (a,b), a,b \in \bar{R}$

Let x_1, x_2 be two different points of (a,b) s.t. $x_1 < x_2$

NB: f dif on $(a,b) \Rightarrow f$ dif on $[x_1,x_2] \Rightarrow f$ is continuous $\Rightarrow$ We can apply the minimum value theorem to f on $[x_1,x_2], \exists x^* \in (x_1,x_2)$ s.t.

$$f'(x^*) = \frac{f(x_2)-f(x_1)}{\underbrace{x_2-x_1}_{>0}} \le 0 \Rightarrow f(x_2)-f(x_1) \le 0 \Rightarrow f(x_2) \le f(x_1)$$

And f is decreasing on (a,b)

Eg $f: R \to R$, $f(x) = x^3 - 3x^2$ $\quad f \in C^1(R)$

$f'(x) = 3x^2 - 6x$.

We suppose that $f'(x) \ge 0$

$3x^2 - 6x \ge 0 \Rightarrow 3x(x-2) \ge 0 \Rightarrow x(x-2) \ge 0 \Rightarrow \begin{cases} x \ge 0 \\ x-2 \ge 0 \end{cases}$ or $\begin{cases} x \le 0 \\ x-2 \le 0 \end{cases} \Rightarrow \left[\begin{array}{l} \begin{cases} x \ge 0 \\ x \ge 2 \end{cases} \\ \begin{cases} x \le 0 \\ x \le 2 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} x \ge 2 \\ x \le 0 \end{array}\right.$

f is increasing on $(-\infty, 0] \cup [2, +\infty)$

By the same logic, f is decreasing on $[0,2]$

f is increasing ...

By the same logic, f is decreasing on [0,2]

Eg $f: R \to R$, $f(x) = x^3$, $f \in C^1(R)$ and $f'(x) = 3x^2$

f is strictly increasing

$f'(x) > 0 \ \forall x \in R \Rightarrow$ f is strictly increasing on R

But f is strictly increasing on $R \not\Rightarrow f'(x) > 0, \forall x \in R$

Proposition

Let f be a dif. function defined on (a, b), with $a, b \in \bar{R}$

$f'(x) > 0, \forall x \in (a, b) \Rightarrow f$ is strictly increasing on (a, b)

Proof

Assume that $f'(x) > 0$ for any $x \in (a, b)$

Let x_1, x_2 be a pair of points of (a, b) with $x_1 < x_2$

f is dif. on $(a, b) \Rightarrow f$ is dif. (and continuous) on x_1 and x_2

$\Rightarrow$ we can apply the mean value theorem to f on $[x_1, x_2]$

$\exists x^* \in (x_1, x_2)$ s.t.

$$f'(x^*) = \frac{f(x_2) - f(x_1)}{\underbrace{x_2 - x_1}_{>0}} > 0 \Rightarrow f(x_2) - f(x_1) > 0 \Rightarrow f(x_2) > f(x_1)$$

And f is strictly increasing on (a, b)

Proposition

Let f be a dif. function defined on (a, b), with $a, b \in \bar{R}$

$f'(x) < 0, \forall x \in (a, b) \Rightarrow f$ is strictly decreasing on (a, b)

Proof

Assume that $f'(x) < 0$ for any $x \in (a, b)$

Let x_1, x_2 be a pair of points of (a, b) with $x_1 < x_2$

f is dif. on $(a, b) \Rightarrow f$ is dif. (and continuous) on x_1 and x_2

$\Rightarrow$ we can apply the mean value theorem to f on $[x_1, x_2]$

$\exists x^* \in (x_1, x_2)$ s.t.

$$f'(x^*) = \frac{f(x_2) - f(x_1)}{\underbrace{x_2 - x_1}_{>0}} < 0 \Rightarrow f(x_2) - f(x_1) < 0 \Rightarrow f(x_2) < f(x_1)$$

And f is strictly increasing on (a, b)

Eg $f: (0, +\infty) \to R$, $f(x) = \frac{\ln x}{x}$ $\quad f \in C^1((0, +\infty))$

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x}$$

$f'(x) \geq 0$: $\frac{1 - \ln x}{x} \geq 0 \Rightarrow 1 - \ln x \geq 0 \Rightarrow \ln x \leq 1 \Rightarrow e^{\ln x} \leq e^1 \Rightarrow x \leq e$

f is increasing on [0, e] and decreasing on $(e, +\infty)$

f is strictly increasing on $(0, e)$ and s. decreasing on $(e, +\infty)$

$$f''(x) = \frac{-\frac{1}{x} \cdot x^2 - (1 - \ln x) \cdot 2x}{x^4} = \frac{-x - 2x(1 - \ln x)}{x^4} = \frac{-1 - 2(1 - \ln x)}{x^3} = \frac{2\ln x - 3}{x^3}$$

Proposition:

$f: A \subseteq R \to R$ and $x_0 \in A$ interior point. Point x_0 is

1. A local maximizer of f on A if $\exists B_\varepsilon(x_0)$ s.t. f is continuous at x_0 and differntiable at $x \in B_\varepsilon(x_0) - \{x_0\}$ with
   $$x_1 < x_0 < x_2 \Rightarrow f'(x_1) \geq 0 \geq f'(x_2)$$
   $\forall x_1, x_2 \in B_\varepsilon(x_0) \cap A.$

   x_0 is strong local maximizer of f on A if the inequalities are strict
2. A local minimizer of f on A if $\exists B_\varepsilon(x_0)$ s.t. f is continuous at x_0 and differntiable at $x \in B_\varepsilon(x_0) - \{x_0\}$ with
   $$x_1 < x_0 < x_2 \Rightarrow f'(x_1) \leq 0 \leq f'(x_2)$$
   $\forall x_1, x_2 \in B_\varepsilon(x_0) \cap A.$

   x_0 is strong local minimizer of f on A if the inequalities are strict

Proof

1)

Let $B_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon)$ be a neighborhood of x_0 with radius $\varepsilon > 0$

Let $x \in (x_0 - \varepsilon, x_0]$ be any point of $x \in B_\varepsilon(x)$

From the mean value theorem, a point $\exists x^* \in (x, x_0)$ s.t.

$$\frac{f(x_0) - f(x)}{\underbrace{x_0 - x}_{>0}} = \underbrace{f'(x^*)}_{\geq 0} \geq 0 \Rightarrow f(x_0) - f(x) \geq 0 \Rightarrow f(x_0) \geq f(x), \forall x \in (x_0 - \varepsilon, x_0]$$

Let $x \in (x_0, x_0 + \varepsilon)$. From the mean value theorem, a point $\exists x^* \in (x_0, x)$ s.t.

$$\frac{f(x_0) - f(x)}{\underbrace{x_0 - x}_{>0}} = \underbrace{f'(x^*)}_{\leq 0} \geq 0 \Rightarrow f(x_0) - f(x) \leq 0 \Rightarrow f(x_0) \leq f(x), \forall x \in [x_0, x_0 + \varepsilon)$$

Therefore, $\forall x \in B_\varepsilon(x_0), f(x) \leq f(x_0)$ and x_0 is a local maximizer of f on A.

2)

Let $B_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon)$ be a neighborhood of x_0 with radius $\varepsilon > 0$

Let $x \in (x_0 - \varepsilon, x_0]$ be any point of $x \in B_\varepsilon(x)$

From the mean value theorem, a point $\exists x^* \in (x, x_0)$ s.t.

$$\frac{f(x_0) - f(x)}{\underbrace{x_0 - x}_{>0}} = \underbrace{f'(x^*)}_{\leq 0} \geq 0 \Rightarrow f(x_0) - f(x) \leq 0 \Rightarrow f(x_0) \leq f(x), \forall x \in (x_0 - \varepsilon, x_0]$$

Let $x \in (x_0, x_0 + \varepsilon)$. From the mean value theorem, a point $\exists x^* \in (x_0, x)$ s.t.

$$\frac{f(x_0) - f(x)}{\underbrace{x_0 - x}_{>0}} = \underbrace{f'(x^*)}_{\geq 0} \geq 0 \Rightarrow f(x_0) - f(x) \geq 0 \Rightarrow f(x_0) \geq f(x), \forall x \in [x_0, x_0 + \varepsilon)$$

Therefore, $\forall x \in B_\varepsilon(x_0), f(x) \geq f(x_0)$ and x_0 is a local maximizer of f on A.

$f: R \to R$ $f(x) = \begin{cases} e^x & x < 0 \\ -x & x \geq 0 \end{cases}$

f is cont. on R and dif. on $R - \{0\}$ ($\lim_{x \to 0^-} \frac{e^x - 1}{x - 0} = 1 = f'_-(0)$; $\lim_{x \to 0^+} \frac{1 - x - 1}{x} = -1 = f'_+(0)$)

$\exists B_\varepsilon(0)$: $\forall x \in B_\varepsilon(0)$

$x_1 < 0 < x_2 \Rightarrow e^{x_1} \geq 0 \geq -1$ $f'(x) = \begin{cases} e^x & x < 0 \\ -1 & x > 0 \end{cases}$

Hence $x_0 = 0$ is a local max. of f on R

The inequalities are strict, hence $x_0 = 0$ is a strong local/global max of f on R

Corollary:

Let $f: A \subseteq R \to R$ and $x_0 \in A$ interior point of A

Point x_0 is said to be:

1. A local maximizer of f on A if $\exists B_\varepsilon(x_0)$ such that f is differentiable with $f'(x_0) = 0$ and
$$x_1 < x_0 < x_2 \Rightarrow f'(x_1) \geq 0 \geq f'(x_2)$$
$\forall x_1, x_2 \in B_\varepsilon(x_0) \cap A$. x_0 is a strong local maximizer of f on A if the inequalities are strict
2. A local minimizer of f on A if $\exists B_\varepsilon(x_0)$ such that f is differentiable with $f'(x_0) = 0$ and
$$x_1 < x_0 < x_2 \Rightarrow f'(x_1) \leq 0 \leq f'(x_2)$$
$\forall x_1, x_2 \in B_\varepsilon(x_0) \cap A$. x_0 is a strong local minimizer of f on A if the inequalities are strict

Eg: $f: R \to R$, $f(x) = x^3 - 3x^2$, $f \in C^1(R)$ and $f'(x) = 3x^2 - 6x$

- $f'(x) = 0 \Rightarrow 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0 \Rightarrow \left[\begin{array}{l} x = 0 \\ x = 2 \end{array}\right. \Rightarrow S = \{0, 2\}$, $\forall x \notin S$, x is not a local min/max of f on R

$x_0 = 0$: $\begin{cases} x_1 < 0 < x_2 \Rightarrow f'(x_1) \geq 0 \geq f'(x_2) \\ \forall x_1, x_2 \in B_\varepsilon(0), \text{ and } x_0 = 0 \text{ is a local max of } f \text{ on } R \text{ and loc. max value } f(0) = 0 \end{cases}$

$x_0 = 2$: $\begin{cases} x_1 < 0 < x_2 \Rightarrow f'(x_1) \leq 0 \leq f'(x_2) \\ \forall x_1, x_2 \in B_\varepsilon(2), \text{ and } x_0 = 2 \text{ is a local min of } f \text{ on } R \text{ \& loc. min value } f(2) = -4 \end{cases}$

As $x \to +\infty$ then $f(x) = x^3 - 3x^2 \sim x^3 \to +\infty$ (No

Corollary:

$f: A \subseteq R \to R$ and $x_0 \in A$ interior point of A.

x_0 is a strong

1. Local maximizer of f on A if $\exists \mathrm{B}_\varepsilon(x_0)$ where f is diff with continuity 2 times at x_0 with $f'(x_0) = 0$ and $f''(x_0) < 0$
2. Local minimizer of f on A if $\exists \mathrm{B}_\varepsilon(x_0)$ where f is diff with continuity 2 times at

Proof

1)

f'' is continuous at x_0, therefore $\lim_{x \to x_0} f''(x) = f''(x_0) < 0$

From the permanence of sign theorem, $\exists B_\varepsilon(x_0)$ s.t.

$f''(x) < 0, \forall x \in B_\varepsilon(x_0)$. f' is strictly decreasing on $B_\varepsilon(x_0)$

From the Mean value theorem of f on $[x_1, x_2]$ with $x_1, x_2 \in B_\varepsilon(x_0), \exists x^* \in (x_1, x_2)$ s.t.

$$f'(x^*) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \Rightarrow f(x_2) \geq f(x_1)$$

and f is increasing on (a, b)

2)

f'' is continuous at x_0, therefore $\lim_{x \to x_0} f''(x) = f''(x_0) > 0$

From the permanence of sign theorem, $\exists B_\varepsilon(x_0)$ s.t.

$f''(x) > 0, \forall x \in B_\varepsilon(x_0)$. f' is strictly increasing on $B_\varepsilon(x_0)$

From the Mean value theorem of f on $[x_1, x_2]$ with $x_1, x_2 \in B_\varepsilon(x_0), \exists x^* \in (x_1, x_2)$ s.t.

$$f'(x^*) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \Rightarrow f(x_2) \leq f(x_1)$$

and f is decreasing on (a, b)

Eg $f: R \to R$, $f(x) = x^4$

$f \in C^2(R)$, $f'(x) = 4x^3$, $f''(x) = 12x^2$

$f'(x) = 0 \Rightarrow x = 0$

$f''(0) = 0$

$\forall x \in R$, $f(0) \leq f(x)$, $x = 0$ is a global min of f on R

Eg $f: R \to R$, $f(x) = x^3 - 3x^2$, $f \in C^2(R)$

$f'(x) = 3x^2 - 3x$, $f''(x) = 6x - 6$

- $f'(x) = 0 \Rightarrow \ldots \Rightarrow \left[\begin{array}{l} x = 0 \\ x = 2 \end{array}\right. \Rightarrow S = \{0, 2\}$

- $f''(0) = 6 \cdot 0 - 6 = -6 < 0 \Rightarrow x = 0$ is a loc. max. of f
- $f''(2) = 12 - 6 = 6 > 0 \Rightarrow x = 2$ is a loc. min. of f

---

NB $f \in C^2(A)$, $A \subseteq R$, $f'(x^*) = 0$ (x^* is a stat point of f)

and $f''(x^*) = 0 \Rightarrow$ no conclusion about the nature of x^*

Proposition:

$f: (a, b) \to R$ differentiable on (a, b) with $a, b \in \bar{R}$

$x_0 \in (a, b)$ is

1. A global maximizer of f on (a, b) if

   $\forall x_1, x_2 \in (a, b), x_1 < x_0 < x_2 \Rightarrow f'(x_1) \geq 0 \geq f'(x_2)$

2. A global minimizer of f on (a, b) if

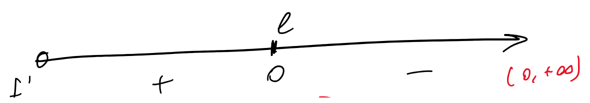
   $\forall x_1, x_2 \in (a, b), x_1 < x_0 < x_2 \Rightarrow f'(x_1) \leq 0 \leq f'(x_2)$

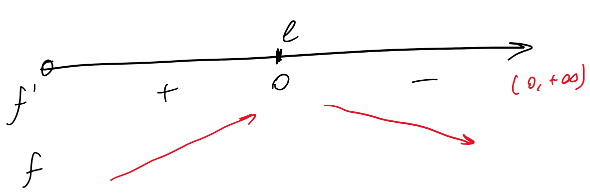
Proof at home

Eg $f: (0, +\infty) \to R$, $f(x) = \frac{\ln x}{x}$; f is dif on $(0, +\infty)$

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

$f'(x) \geq 0 \quad \frac{1 - \ln x}{x^2} \geq 0 \Rightarrow 1 - \ln x \geq 0 \Rightarrow \ln x \leq 1 \Rightarrow \ln x \leq \ln e \Rightarrow x \leq e$





De l'Hôpital rule

We use this to calculate limits with indefinite forms $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \begin{cases}\frac{0}{0}\\ \frac{\infty}{\infty}\end{cases}$$

Theorem De l'Hôpital rule

$f, g: (a,b) \to R$ differentiable on (a,b) with $a, b \in \bar{R}$ and $g'(x) \neq 0\ \forall x \in (a,b)$ and

$$\lim_{x\to x_0}\frac{f'(x)}{g'(x)} = L \in \bar{R} \text{ with } x_0 \in [a,b]$$

If either $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ or $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \pm\infty$ then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = L$$

Side note: Limit of the difference of derivatives is the same as the difference of the original function. If it doesn't work with f', we just take f'', f''' and so on

Eg: $\lim_{x\to 0}\frac{x\cdot\sin x}{e^{x^2}-1}$ Ind. form $\frac{0}{0}$

$f(x) = x\cdot\sin x$, $f'(x) = \sin x + x\cdot\cos x$

$g(x) = e^{x^2}-1$, $g'(x) = 2x e^{x^2}$

$\lim_{x\to 0}\frac{\sin x + x\cos x}{2x e^{x^2}}$ $\frac{0}{0}$

$\lim_{x\to 0}\frac{\cos x + \cos x - x\cdot\sin x}{2[e^{x^2} + x 2x\cdot e^{x^2}]} = 1 = L \Rightarrow \lim_{x\to 0}\frac{x\cdot\sin x}{e^{x^2}-1} = 1$

Eg: $f \mid \lim_{x\to 0}\frac{x^4}{x^2+2\cos x-2}$ Ind. form $\frac{0}{0}$

$f' \mid \lim_{x\to 0}\frac{4x^3}{2x-2\sin x} \to \frac{0}{0}$

$f'' \mid \lim_{x\to 0}\frac{12x^2}{2-2\cos x} \to \frac{0}{0}$

$f^{(3)} \mid \lim_{x\to 0}\frac{24x}{2\sin x} \to \frac{0}{0}$

$f^{(4)} \mid \lim_{x\to 0}\frac{24}{2\cos x} \to 12 = L$

$\lim_{x\to 0}\frac{x^4}{x^2+2\cos x-2} = 12$

Chapter 29: Taylor's polynomial approximation

lunedì 4 dicembre 2023 11:32

$f:(a,b) \to R$ dif at $x_0 \in (a,b)$, then $\exists B_\varepsilon(x_0)$ where we can approximate G_f at x_0 with

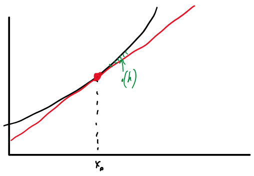
$$f(x_0+h) = f(x_0) + f'(x_0)\cdot h + o(h) \text{ as } h \to 0$$

error of approximation

Or

$$f(x) = f(x_0) + f'(x_0)\cdot(x - x_0) + o(x - x_0) \text{ as } x \to x_0$$

Linear approximation of f at x_0



Target:

we want to approximate f in a neighborhood of x_0 using a polynomial function with degree n

$n = 1 \to$ Linear approximation

$n = 2 \to$ Quadratic approximation

Definition

$f:(a,b) \to R$ has a polynomial expansion of degree at $x_0 \in (a,b)$

if a polynomial function $p_n: R \to R$ exists with degree at most n such that

$$f(x_0+h) = p_n(h) + o(h) \text{ as } h \to 0$$

With $x_0 + h \in (a,b)$

NB

Linear polynomial $p_1: R \to R, p_1(h) = \alpha_0 + \alpha_1 h$

$$f(x_0+h) = \alpha_0 + \alpha_1 h + o(h), h \to 0$$

Quadratic approximation $p_2: R \to R, p_2(h) = \alpha_0 + \alpha_1 h + \alpha_2 h^2$

$$f(x_0+h) = \alpha_0 + \alpha_1 h + \alpha_2 h^2 + o(h)$$

Lemma:

$f:(a,b) \to R$ has at most one polynomial expansion of degree n at each point $x_0 \in (a,b)$

Def:

$f:(a,b) \to R$ n times differentiable at $x_0 \in (a,b)$. The polynomial $T_n: R \to R$ of degree at most n given by

$$T_n(h) = f(x_0) + f'(x_0)\cdot h + \frac{1}{2!}f''(x_0)\cdot h^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)\cdot h^n$$

$\equiv$

$$T_n(h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\cdot h^k$$

Is called the Taylor polynomial of degree n of f at x_0

NB

1. $f^{(0)} = f$

$x_0 = 0 \Rightarrow$ The Taylor's polynomial is called Maclaurin's polynomial

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\cdot h^k$$

Eg: $f: R \to R$, $f(x) = x\cdot e^x$, $x_0 = 0$ $\quad f \in C^2(R)$ $\quad f(0) = 0\cdot e^0 = 0$

$f'(x) = 1\cdot e^x + x\cdot e^x = (1+x)e^x \Rightarrow (1+0)e^0 = 1$

$f''(x) = 1\cdot e^x + (1+x)e^x = (2+x)e^x \Rightarrow (2+0)e^0 = 2$

$T_2(h) = f(x_0) + f'(x_0)\,h + \frac{1}{2!}f''(x_0)\,h^2$

$\underset{x_0=0}{=} 0 + 1\cdot h + \frac{1}{2}\,2 = x + x^2 \quad T_2(x) = x + x^2$

NB

$$x_0 = 0, h = x$$

$$T_n(x) = f(0) + f'(0) * x + \frac{1}{2!}f''(0)x^2 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n$$

Eg: $f: R \to R$

$f(x) = \sqrt{1+x} = (1+x)^{1/2}$, $x_0 = 0$

$T_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$

$f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \Rightarrow f'(0) = \frac{1}{2}$

$\ldots^{-3/2} \quad f''(0) = -\frac{1}{4}(1)^{-3/2} = -\frac{1}{4}$

$f(0)=1$

$f'(x)=\frac{1}{2}(1+x)^{-1/2} \Rightarrow f'(0)=\frac{1}{2}$

$f''(x)=\frac{1}{2\sqrt{1+x}}=\frac{1}{2}\cdot\left(-\frac{1}{2}\right)(1+x)^{-3/2} \Rightarrow f''(0)=-\frac{1}{4}(1)^{-3/2}=-\frac{1}{4}$

$T_2(x)=1+\frac{1}{2}x+\frac{1}{2!}\cdot\left(-\frac{1}{4}\right)x^2=\boxed{1+\frac{x}{2}-\frac{x^2}{8}}$

Theorem: (Taylor-Peano)

$f:(a,b)\to R$ n times differntiable at $x_0\in(a,b)$.

Function f has at x_0 a unique polynomial expansion p_n of degree n given by

$p_n(h)=T_n(h)$

Proof for n=2

Let $x_0\in(a,b)$

Assume f is twice differentiable on (a,b)

Let ϕ be a real function defined on (a,b) given by

$$\phi:(a,b)\to R, \phi(h)=f(x_0+h)-f(x_0)-f'(x_0)\cdot h-\frac{1}{2!}f''(x_0)\cdot h^2$$

That is

$$\phi(h)=o(h^2)$$

And we want to show that

$$\lim_{h\to 0}\frac{\phi(h)}{h^2}=0$$

Function ϕ is differentialbe and its derivative is

$$\phi'(h)=f'(x_0+h)-0-f'(x_0)-\frac{1}{2}\cdot f''(x_0)\cdot 2h$$
$$=f'(x_0+h)-f'(x_0)-hf''(x_0)$$

f and f' are continuous at x_0, hence ϕ and ϕ'' are continuous

$$\lim_{h\to 0}\phi(h)=\phi(0)=0 \text{ and } \lim_{h\to 0}\phi'(h)=\phi'(0)=0$$

Using the de l'Hôpital theorem, we have:

$$\lim_{h\to 0}\frac{\phi'(h)}{2h}=L\Rightarrow\lim_{h\to 0}\frac{\phi(h)}{h^2}=L$$

We have

$$\lim_{h\to 0}\frac{\phi'(h)}{2h}=\lim_{h\to 0}\left[\frac{f'(x_0+h)-f'(x_0)-h\cdot f''(x_0)}{2h}\right]=\frac{1}{2}\lim_{h\to 0}\left[\frac{f'(x_0+h)-f'(x_0)}{h}-\frac{h\cdot f''(x_0)}{h}\right]=$$
$$=\frac{1}{2}\lim_{h\to 0}\left[\frac{f'(x_0+h)-f'(x_0)}{h}-f''(x_0)\right]=\frac{1}{2}[f''(x_0)-f''(x_0)]=0$$

We have

$$\lim_{h\to 0}\frac{\phi'(h)}{2h}=0\in R\Rightarrow\lim_{h\to 0}\frac{\phi(h)}{h^2}=0$$

NB

From this theorem, we have:

$$f(x_0+h)=T_n(h)+o(h^n)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}\cdot h^k+0(h^n)$$

Important Maclaurin's Formulas

1. $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+o(x^n)\equiv\sum_{k=0}^{n}\frac{x^k}{k!}+o(x^k)$
2. $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+o(x^{2n+1})\equiv\sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}+o(x^{2n+1})$
3. $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+o(x^{2n})\equiv\sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}+o(x^{2n})$
4. $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots+(-1)^{n+1}\frac{x^n}{n}+o(x^n)\equiv\sum_{k=0}^{n}(-1)^k\frac{x^{k+1}}{k+1}+o(x^n)$

Substitution principle:

$$e^{2x}=1+(2x)+\frac{(2x)^2}{2!}+o(x^2)=1+2x+2x^2+o(x^2)$$

$$e^{x^2}=1+x^2+\frac{(x^2)^2}{2!}+o\left[(x^2)^2\right]=1+x^2+\frac{x^4}{2}+o(x^4)$$

Proposition

$f:A\subseteq R\to R$ n times differentiable at an interior point $x_0\in A$ with $f^{(k)}(x_0)=0$ for any $1\le k\le n-1$ and $f^{(n)}\neq 0$

We have:

1. In n is **even** and $f^{(n)}(x_0)<0$ then x_0 is a strong local maximizer of f on A
2. In n is **even** and $f^{(n)}(x_0)>0$ then x_0 is a strong local minimizer of f on A
3. In n is odd then x_0 is not a local max/min of f on A; f is $\begin{cases}\text{increasing at } x_0(\textit{locally increasing})\text{if } f^{(n)}>0\\ \text{decreasing at } x_0(\textit{locally decreasing})\text{if } f^{(n)}<0\end{cases}$

(Eg) $f:R\to R$, $f(x)=x^4$, $f\in C^\infty(R)$

(Eg) $f:\mathbb{R}\to\mathbb{R},\ f(x)=x^4,\ f\in C^n(\mathbb{R})$

$f'(x)=4x^3;\ f'(0)=0 \Rightarrow x=0$

$f''(x)=12x^2,\ f''(0)=0$

$f^{(3)}(x)=24x,\ f^{(3)}(0)=0$

$f^{(4)}(x)=24$ and $f^{(4)}(0)=24>0$

Hence $x_0=24$ is a strong local/global min. of f on $\mathbb{R}$ with local/glob. min value of $f(0)=0$

(Eg) $f:\mathbb{R}\to\mathbb{R},\ f(x)=x^3,\ f\in C^n(\mathbb{R})$

$f'(x)=3x^2,\ f'(x)=0 \Rightarrow x=0$

$f''(x)=6x,\ f''(0)=0$

$f^{(3)}(x)=6,\ f^{(3)}(0)=6>0$

x_0 isn't a local max/min of f on $\mathbb{R}$

f is increasing at $x_0=0$

$f:\mathbb{R}\to\mathbb{R},\ f(x)=x^4-4x^3,\ f\in C^n(\mathbb{R})$

$f'(x)=4x^3-12x^2=4x^2(x-3)$

$f'(x)=0 \Rightarrow \left[\begin{array}{l} x=0 \\ x=3 \end{array}\right. \Rightarrow S=\{0,3\}$

$f''(x)=12x^2-24x$

$f''(0)=0$

$f''(3)=36>0 \Rightarrow x_0=3$ is a strong local minimizer of f on $\mathbb{R}$ with local value $f(3)=3^4-4\cdot3^3=-27$

$f^{(3)}(x)=24x-24,\ f^{(3)}(0)=-24<0$

$x=0$ isn't a local max/min of f on $\mathbb{R}$ and f is decreasing at $x_0=0$

https://bit.ly/Peer2PeerBocconi

https://www.blabbocconi.it/it/dispense/

https://bit.ly/Blab_on_Insta

For doubts or suggestions on the handout:

Dzianis Rabchuk

+39 328 705 1367

@_dzianis_ka

For info about our Teaching Division:

PIETRO VILLA

+39 346 2100003

@pietro_villa__

CHIARA TUA

+39 347 9789059

@chiara_tua

Blab
UNILAB NETWORK

Our partners:

DELIVERY VALLEY
NO GENDER KITCHEN

700+
CLUB