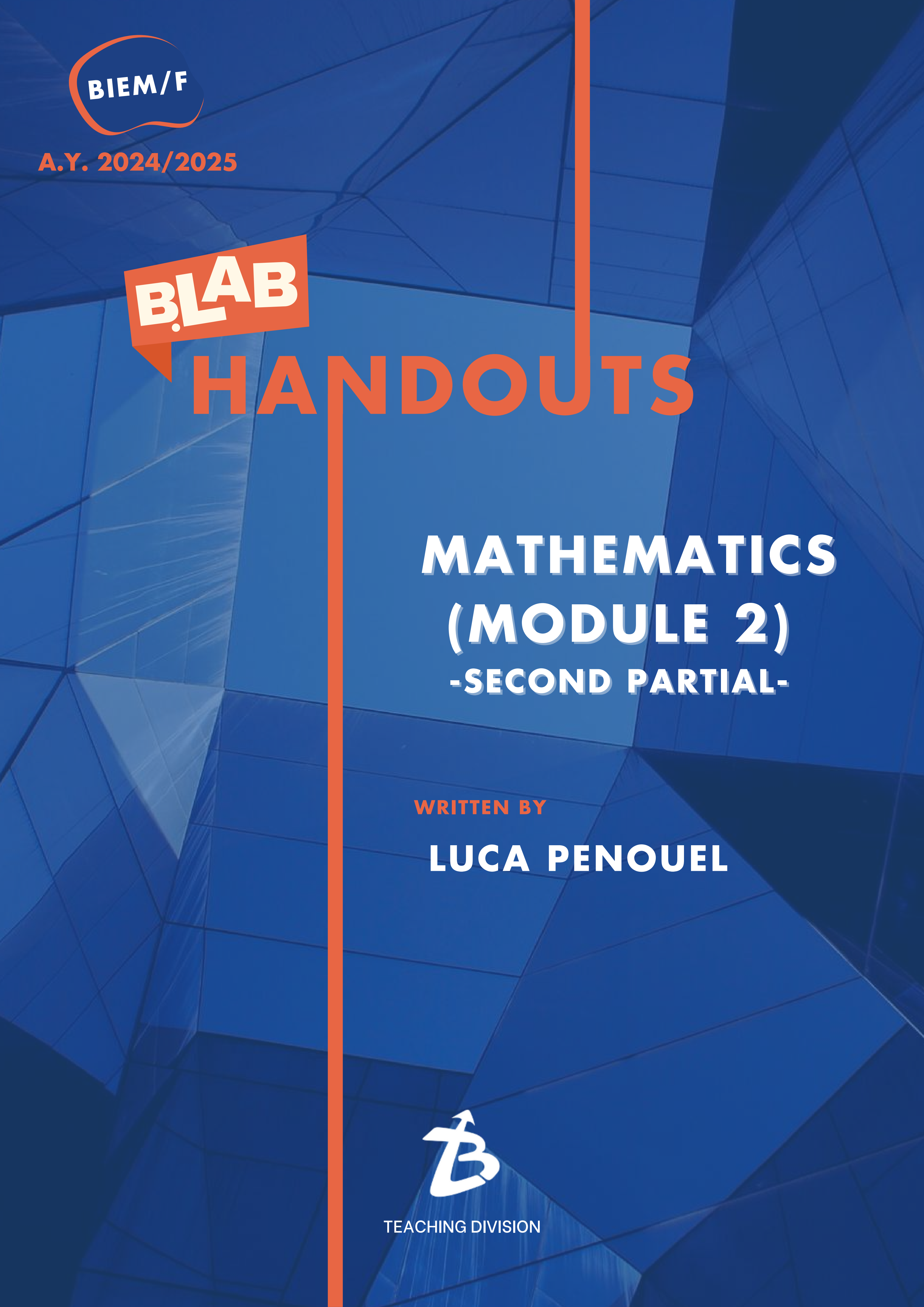

BIEM/F

A.Y. 2024/2025

B.LAB

HANDOUTS

MATHEMATICS (MODULE 2)

-SECOND PARTIAL-

WRITTEN BY

LUCA PENOUEL

TEACHING DIVISION

Probability Calculus

Proofs, Definitions and Theorems

These notes have been compiled by **Luca Penouel** as an additional resource to aid BIEM and BIEF students of the **Mathematics 2 course** (30063).

You will find the proofs, definitions, and theorems listed in the official syllabus for the academic year 2024-2025. The versions of the proofs/definitions/theorems contained herein may differ from those found in the official course textbook; they are adaptations that combine elements from multiple sources to enhance clarity and understanding, while (hopefully) maintaining mathematical correctness.

Please note that this handout IS NOT intended to replace the official course materials. It is provided merely as a supplementary tool to assist in your study and understanding of the course content.

The notes are not in its final form; updates and corrections will be applied as needed. If you notice any errors, kindly contact me privately so that I can make the necessary revisions.

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Updated: April 25, 2025

* = IN CLASS
H = HOMEWORK

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PROOFS

1) MONOTONICITY OF MEASURES* (2043)

STATEMENT: EVERY MEASURE IS MONOTONE

PROOF: CONSIDER TWO SUBSETS A AND B OF Ω WITH $A \subseteq B$.
DEFINE $C = B - A$, SINCE $A \subseteq B$ WE KNOW $B = C \cup A$ AND $C \cap A = \emptyset$
SINCE μ IS ADDITIVE AND POSITIVE WE HAVE:
$\mu(B) = \mu(C \cup A) = \mu(C) + \mu(A) \geq \mu(A)$, SO $\mu(B) \geq \mu(A)$.

2) TOTAL PROBABILITY* (2045)

STATEMENT: CONSIDER A MEASURE $\mu : 2^{\Omega} \to [0, +\infty)$, IT HOLDS:
$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) \quad \forall A, B \subseteq \Omega$

PROOF: LET $A, B \subseteq \Omega$, WE HAVE: $A = (A-B) \cup (A \cap B)$ AND $(A-B) \cap (A \cap B) = \emptyset$
SINCE μ IS ADDITIVE: $\mu(A) = \mu(A-B) + \mu(A \cap B)$, SO $\mu(A-B) = \mu(A) - \mu(A \cap B)$

SIMILARLY BY INTERCHANGING A AND B: $\mu(B-A) = \mu(B) - \mu(A \cap B)$

NOW CONSIDER $A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$, WHICH ARE THREE PAIRWISE DISJOINT SETS, SO BY ADDITIVITY:

$$\mu(A \cup B) = \mu(A-B) + \mu(A \cap B) + \mu(B-A)$$
$$\mu(A \cup B) = \mu(A) - \mu(A \cap B) + \mu(A \cap B) + \mu(B) - \mu(A \cap B)$$
$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

3) PROBABILITY OF THE COMPLEMENT^{H} (2052)

STATEMENT: GIVEN A PROBABILITY MEASURE $P: 2^{\Omega} \to [0,1]$, $\forall A \in 2^{\Omega}$ WE HAVE : $P(A^c) = 1 - P(A)$

PROOF: SINCE $A \cup A^c = \Omega$ AND $A \cap A^c = \emptyset$ BY ADDITIVITY: $P(A \cup A^c) = P(A) + P(A^c)$ AND SINCE $P(A \cup A^c) = 1$ IT BECOMES $1 = P(A) + P(A^c) \implies P(A^c) = 1 - P(A)$

4) SIMPLE PROBABILITIES AS SUM OF DIRAC PROBABILITIES* (2058)

STATEMENT: LET $P: 2^{\Omega} \to [0,1]$ BE A SIMPLE PROBABILITY MEASURE FOR EACH EVENT A : $P(A) = \sum_{\omega \in \operatorname{supp} P} P(\omega)\,\delta_{\omega}(A)$

PROOF: IT IS ENOUGH TO SHOW THAT : $\sum_{\omega \in \operatorname{supp} P} P(\omega) = \sum_{\omega \in \operatorname{supp} P} P(\omega)\,\delta_{\omega}(A)$

, SINCE WE KNOW THAT $P(A) = \sum_{\omega \in A \cap \operatorname{supp} P} P(\omega)$, THEN IT FOLLOWS THAT:

$P(A) = \sum_{\omega \in \operatorname{supp} P} P(\omega)\,\delta_{\omega}(A)$ AS DESIRED.

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5) SIMPLE PROBABILITIES ARE COUNTABLY ADDITIVE *

(2062)

STATEMENT: SIMPLE PROBABILITIES ARE COUNTABLY ADDITIVE.

PROOF: i) WE BEGIN BY PROVING THAT A DIRAC PROBABILITY δ_{ω_0} WITH $\omega_0 \in \Omega$ IS COUNTABLY ADDITIVE. CONSIDER A COLLECTION OF EVENTS $\{A_m\}$, WITH $A_m \downarrow A$. WE HAVE TWO CASES: $\omega_0 \in A$ OR $\omega_0 \notin A$. IN THE FIRST CASE WE HAVE $\omega_0 \in A_m$ $\forall m \in \mathbb{N}$, AND IN PARTICULAR $\delta_{\omega_0}(A_m) = 1$ $\forall m \geq 1$, PROVING THAT $\lim_m \delta_{\omega_0}(A_m) = 1 = \delta_{\omega_0}(A)$. IN THE SECOND CASE $\exists\ \bar{m} \geq 1$ SUCH THAT $\omega_0 \notin A_m$ $\forall m \geq \bar{m}$ AND IN PARTICULAR $\delta_{\omega_0}(A_m) = 0$ $\forall m \geq \bar{m}$, PROVING THAT $\lim_m \delta_{\omega_0}(A_m) = 0 = \delta_{\omega_0}(A)$. WE CONCLUDE THAT δ_{ω_0} IS COUNTABLY ADDITIVE.

ii) NOW LET $P: 2^{\Omega} \to [0,1]$ BE A SIMPLE PROBABILITY. WE KNOW THAT $P(A) = \sum_{\omega \in A \cap \operatorname{supp} P} P(\omega)$ $\forall A \subseteq \Omega$ WHICH BECOMES: $P(A) = \sum_{\omega \in \operatorname{supp} P} P(\omega)\delta_\omega(A)$ $\forall A \subseteq \Omega$

NOW TAKE A COUNTABLE COLLECTION OF EVENTS $\{A_m\}$ WITH $A_m \downarrow A$. SINCE EACH DIRAC PROBABILITY δ_ω IS COUNTABLY ADDITIVE WE HAVE:

$$\lim_m P(A_m) = \lim_m \sum_{\omega \in \operatorname{supp} P} P(\omega)\delta_\omega(A) = \sum_{\omega \in \operatorname{supp} P} P(\omega) \lim_m \delta_\omega(A_m) =$$

$$= \sum_{\omega \in \operatorname{supp} P} P(\omega)\delta_\omega(A) = P(A)$$

SO, SINCE $\lim_m P(A_m) = P(A)$, P IS COUNTABLY ADDITIVE.

6) CHARACTERIZATION OF SIMPLE PROBABILITIES* (2070)

STATEMENT: LET $P: 2^{\Omega} \to [0,1]$ BE A SIMPLE PROBABILITY. TWO RANDOM VARIABLES $f, g : \Omega \to \mathbb{R}$ ARE EQUAL P-a.e. $\Longleftrightarrow$ $\forall\ \omega \in \operatorname{supp} P$ WE HAVE $f(\omega) = g(\omega)$

PROOF: "IF": LET $f(\omega) = g(\omega)$ $\forall\, \omega \in \operatorname{supp} P$. THEN $\operatorname{supp} P \subseteq (f = g)$ AND SO, BY THE MONOTONICITY OF P WE HAVE $P(f=g) = 1$.

"ONLY IF": SUPPOSE $P(f=g) = 1$, THEN WE KNOW THAT $\operatorname{supp} P \subseteq (f=g)$. THEREFORE $f(\omega) = g(\omega)$ $\forall\, \omega \in \Omega$

7) SAME EXPECTED VALUE* (2073)

STATEMENT: LET $P: 2^{\Omega} \to [0,1]$ BE A SIMPLE PROBABILITY. IF TWO RANDOM VARIABLES $f, g : \Omega \to \mathbb{R}$ ARE EQUAL P-a.e. THEN $E_P(f) = E_P(g)$

PROOF: SINCE f, g ARE EQUAL P-a.e. WE HAVE THAT $f(\omega) = g(\omega)$ $\forall\, \omega \in \operatorname{supp} P$, THEREFORE WE CONCLUDE THAT:

$$E_P(f) = \sum_{\omega \in \operatorname{supp} P} f(\omega) P(\omega) = \sum_{\omega \in \operatorname{supp} P} g(\omega) P(\omega) = E_P(g)$$

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8/9/10) PROPERTIES OF EXPECTED VALUE (2074)

STATEMENT: LET $P: 2^{\Omega} \to [0,1]$ BE A SIMPLE PROBABILITY. FOR ALL RANDOM VARIABLES $f, g: \Omega \to \mathbb{R}$ WE HAVE:

i) $E_P(\alpha f + \beta g) = \alpha E_P(f) + \beta E_P(g) \quad \forall \alpha, \beta \in \mathbb{R}$

ii) $E_P(f) \geq E_P(g)$ IF $f \geq g$

iii) $E_P(f) = \sum_{\omega \in A} f(\omega) P(\omega)$ IF EVENT A IS FINITE AND CONTAINS SUPP P.

PROOF:

i) $\forall \alpha, \beta \in \mathbb{R}$ WE HAVE: $E_P(\alpha f + \beta g) = \sum_{\omega \in \text{supp} P} (\alpha f(\omega) + \beta g(\omega)) \cdot P(\omega)$
$= \alpha \sum_{\omega \in \text{supp} P} f(\omega) P(\omega) + \beta \sum_{\omega \in \text{supp} P} g(\omega) P(\omega) = \alpha E_P(f) + \beta E_P(g)$

ii) LET $f \geq g$ i.e. $f(\omega) \geq g(\omega)$ $\forall \omega \in \Omega$. AS $P \geq 0$ WE HAVE THAT $f(\omega) P(\omega) \geq g(\omega) P(\omega)$ $\forall \omega \in \Omega$ WHICH IN TURN IMPLIES:

$$\sum_{\omega \in \text{supp} P} f(\omega) P(\omega) \geq \sum_{\omega \in \text{supp} P} g(\omega) P(\omega) \implies E_P(f) \geq E_P(g)$$

iii) LET A BE A FINITE EVENT WITH $\text{supp} P \subseteq A$. WE KNOW THAT $P(\omega) = 0$ $\forall \omega \in A$ WITH $\omega \notin \text{supp} P$, SO:

$$\sum_{\omega \in \text{supp} P} f(\omega) P(\omega) = \sum_{\omega \in A} f(\omega) P(\omega) \implies E_P(f) = \sum_{\omega \in A} f(\omega) P(\omega).$$

11/12) PROPERTIES OF VARIANCE (2080)

STATEMENT: LET $P: 2^{\Omega} \to \mathbb{R}$ BE A SIMPLE PROBABILITY.
$\forall$ RANDOM VARIABLE $f: \Omega \to \mathbb{R}$ IT HOLDS:

i) $V_P(f) = E_P(f^2) - E_P(f)^2$

ii) $V_P(\alpha f + \beta) = \alpha^2 V_P(f) \quad \forall \alpha, \beta \in \mathbb{R}$

PROOF:

i) FOR EACH $\omega \in \Omega$: $(f(\omega) - E_P(f))^2 = f(\omega)^2 - 2f(\omega)E_P(f) + E_P(f)^2$
AND BY DEFINITION OF COVARIANCE AND LINEARITY OF THE EXPECTED VALUE:

$$V_P(f) = E_P(f - E_P(f))^2$$
$$(1) \quad = E_P(f^2 - 2E_P(f)f + E_P(f))$$
$$(2) \quad = E_P(f^2) - 2E_P(f) \cdot E_P(f) + E_P(f)^2$$
$$(3) \quad = E_P(f^2) - E_P(f)^2$$

AS DESIRED.

ii) BY THE LINEARITY OF THE EXPECTED VALUE, $\forall \omega \in \Omega$ WE HAVE:

$(\alpha f(\omega) + \beta - E_P(\alpha f + \beta))^2 = \alpha^2 (f(\omega) - E_P(f))^2$, WHICH BECOMES:

$$V_P(\alpha f + \beta) = E_P(\alpha f + \beta - E_P(\alpha f + \beta))^2 =$$
$$(1) \quad = E_P(\alpha^2 (f - E_P(f))^2)$$
$$(2) \quad = \alpha^2 E_P(f - E_P(f))^2$$
$$(3) \quad = \alpha^2 V_P(f)$$

AS DESIRED.

13/14) PROPERTIES OF COVARIANCE (2082)

STATEMENT: LET $P: 2^{\Omega} \to \mathbb{R}$ BE A SIMPLE PROBABILITY. $\forall$ RANDOM VARIABLE $f, g: \Omega \to \mathbb{R}$ IT HOLDS:

i) $\mathrm{COV}_P(f,g) = E_P(fg) - E_P(f)E_P(g)$

ii) $\mathrm{COV}_P(\alpha f + \beta, \gamma g + \delta) = \alpha\gamma\, \mathrm{COV}_P(f,g) \quad \forall \alpha, \beta, \gamma, \delta \in \mathbb{R}$

PROOF:

i) $\forall \omega \in \Omega$:

$$(f(\omega) - E_P(f))(g(\omega) - E_P(g)) = f(\omega)g(\omega) - f(\omega)E_P(g) - E_P(f)g(\omega) + E_P(f)E_P(g)$$

AND BY DEFINITION OF COVARIANCE AND LINEARITY OF THE EXPECTED VALUE:

$$\begin{aligned} \mathrm{COV}(f,g) &= E_P(fg - E_P(g)f - E_P(f)g + E_P(f)E_P(g)) \\ &\overset{(1)}{=} E_P(fg) - 2E_P(f)E_P(g) + E_P(f)E_P(g) \\ &\overset{(2)}{=} E_P(fg) - E_P(f)E_P(g) \end{aligned}$$

AS DESIRED.

ii) BY THE LINEARITY OF THE EXPECTED VALUE, $\forall \omega \in \Omega$ WE HAVE:

$$(\alpha f(\omega) + \beta - E_P(\alpha f + \beta))(\gamma g(\omega) + \delta - E_P(\gamma g + \delta)) = \alpha\gamma(f(\omega) - E_P(f))(g(\omega) - E_P(g))$$

BY DEFINITION OF COVARIANCE WE HAVE:

$$\begin{aligned} \mathrm{COV}_P(\alpha f + \beta, \gamma g + \delta) &= E_P((\alpha f + \beta - E_P(\alpha f + \beta))(\gamma g + \delta - E_P(\gamma g + \delta))) \\ &\overset{(1)}{=} E_P(\alpha\gamma(f - E_P(f))(g - E_P(g))) \\ &\overset{(2)}{=} \alpha\gamma\, \mathrm{COV}_P(f,g) \end{aligned}$$

15) VARIANCE OF A SUM* (2083)

STATEMENT: LET $P: 2^{\Omega} \to \mathbb{R}$ BE A SIMPLE PROBABILITY. $\forall$ RANDOM VARIABLE $f, g: \Omega \to \mathbb{R}$ IT HOLDS:

$$V_P(f+g) = V_P(f) + V_P(g) + 2\,\text{Cov}_P(f,g)$$

PROOF: BY THE LINEARITY OF THE EXPECTED VALUE AND BY

$$\begin{aligned}
V_P(f+g) &= E_P(f+g-E_P(f+g))^2 \\
\overset{(1)}{} &= E_P((f-E_P(f)) + (g-E_P(g)))^2 \\
\overset{(2)}{} &= E_P(f-E_P(f))^2 + E_P(g-E_P(g))^2 + 2E_P(f-E_P(f))(g-E_P(g)) \\
\overset{(3)}{} &= V_P(f) + V_P(g) + 2\,\text{cov}_P(f,g) \quad \text{AS DESIRED.}
\end{aligned}$$

16) COVARIANCE IS BOUNDED* (2084)

STATEMENT: LET $P: 2^{\Omega} \to \mathbb{R}$ BE A SIMPLE PROBABILITY. $\forall$ RANDOM VARIABLES $f, g: \Omega \to \mathbb{R}$: $|\text{cov}_P(f,g)| \leq \sigma_P(f)\,\sigma_P(g)$

PROOF: LET $\text{supp}\,P = \{\omega_1 \dots \omega_m\}$ BE THE SUPPORT OF THE SIMPLE PROBABILITY P. FIRST, ASSUME THAT $E_P(f) = E_P(g) = 0$. $\forall\, i = 1, 2 \dots m$ SET: $x_i := f(\omega_i)\sqrt{P(\omega_i)}$ AND $y_i := g(\omega_i)\sqrt{P(\omega_i)}$ AND BY THE CAUCHY SWARTZ INEQUALITY:

$$\begin{aligned}
|\text{cov}_P(f,g)| &= \Big|\sum_{\omega \in \text{supp}P} f(\omega)\,g(\omega)\,P(\omega)\Big| = \Big|\sum_{i=1}^{m} x_i y_i\Big| = |x \cdot y| \leq \|x\|\,\|y\| \\
&= \sqrt{\sum_{i=1}^{m} (x_i\sqrt{P_i})^2}\ \sqrt{\sum_{i=1}^{m} (y_i\sqrt{P_i})^2} = \sqrt{\sum_{i=1}^{m} x_i^2 P_i}\ \sqrt{\sum_{i=1}^{m} y_i^2 P_i} = \\
&= \sqrt{\sum_{\omega \in \text{supp}P} f^2(\omega)P(\omega)}\ \sqrt{\sum_{\omega \in \text{supp}P} g^2(\omega)P(\omega)} = \sigma_P(f)\,\sigma_P(g)
\end{aligned}$$

THIS PROVES THE THESIS WHEN $E_P(f) = E_P(g) = 0$. IN THE GENERAL CASE, SET $\tilde{f} = f - E_P(f)$ AND $\tilde{g} = g - E_P(g)$, AS $E_P(\tilde{f}) = E_P(\tilde{g}) = 0$, BY WHAT WE JUST PROVED WE HAVE: $|\text{cov}_P(\tilde{f},\tilde{g})| \leq \sigma_P(\tilde{f})\,\sigma_P(\tilde{g})$.
BY PROPERTIES OF VARIANCE AND COVARIANCE: $\sigma_P(\tilde{f}) = \sigma_P(f)$, $\sigma_P(\tilde{g}) = \sigma_P(g)$ AND $\text{cov}_P(\tilde{f},\tilde{g}) = \text{cov}_P(f,g)$. WE CONCLUDE THAT THE THESIS HOLDS $\forall$ RANDOM VARIABLE $f, g: \Omega \to \mathbb{R}$.

17) DISTRIBUTION FUNCTION IS INCREASING* (2088)

STATEMENT: THE DISTRIBUTION FUNCTION $\Phi : \mathbb{R} \to [0,1]$ IS INCREASING.

PROOF: LET $x, y \in \mathbb{R}$ WITH $y \geq x$, CLEARLY $(f \leq x) \subseteq (f \leq y)$, SINCE P IS MONOTONE THIS IMPLIES: $\Phi(x) = P(f \leq x) \leq P(f \leq y) = \Phi(y)$ SHOWING THAT Φ IS INCREASING.

18) DISTRIBUTION FUNCTIONS OF ESSENTIALLY* BOUNDED R.V.S. (2093)

STATEMENT: IF $f : \Omega \to \mathbb{R}$ IS ESSENTIALLY BOUNDED, $\exists$ SCALARS a AND b SUCH THAT: $\Phi(x) = 0 \ \forall x \leq a$ AND $\Phi(x) = 1 \ \forall x \geq b$

PROOF: LET f BE ESSENTIALLY BOUNDED, BY DEFINITION $\exists\, m, M \in \mathbb{R}$ WITH $P(m \leq f \leq M) = 1$. NOW LET $x < m$, WE HAVE: $(f \leq x) \cap (m \leq f \leq M) = \emptyset$ AND SO $\Phi(x) = P(f \leq x) = 0$. NOW LET $x \geq M$, WE HAVE: $(m \leq f \leq M) \subseteq (f \leq x)$ AND SO, BY THE MONOTONICITY OF P WE HAVE:

$$1 \geq \Phi(x) = P(f \leq x) \geq P(m \leq f \leq M) = 1$$

SO, $\Phi(x) = 1$. NOW IF WE SET $b = M$ AND TAKE ANY $a < m$, IT HOLDS THAT $\Phi(x) = 0 \ \forall x \leq a$ AND $\Phi(x) = 1 \ \forall x \geq b$, AS DESIRED.

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19) CONTINUOUS DENSITY FUNCTIONS PROPERTY 1^* (2097)

STATEMENT: LET Φ BE A DISTRIBUTION FUNCTION WITH A CARRIER $[a,b]$. IF Φ HAS A CONTINUOUS DENSITY FUNCTION ϕ, THEN:

$$\phi(x) = 0 \quad \forall x \notin [a,b]$$

PROOF: BY DEFINITION, $\Phi(x) = \int_{-\infty}^{x} \phi(t)\,dt \ \forall x \in \mathbb{R}$. NOW SET $z \geq b$. SINCE $\int_{-\infty}^{+\infty} \phi(x)\,dx = \int_{a}^{b} \phi\,dx = 1$, WE HAVE $\int_{b}^{z} \phi(x)\,dx = 0 \ \forall z \geq b$. WE KNOW $\phi(t) = 0 \ \forall b \leq t \leq z$. SINCE z WAS CHOSEN ARBITRARILY, WE CONCLUDE THAT $\phi(z) = 0 \ \forall z \geq b$. A SIMILAR ARGUMENT SHOWS THAT $\phi(z) = 0$ $\forall z \leq a$.

20) CONTINUOUS DENSITY FUNCTIONS PROPERTY 2^* (2098)

STATEMENT: A DISTRIBUTION FUNCTION Φ WITH A CARRIER $[a,b]$ HAS A UNIQUE CONTINUOUS DENSITY FUNCTION ϕ $\Longleftrightarrow$, IT IS CONTINUOUSLY DIFFERENTIABLE AND IN THIS CASE: $\Upsilon = \Phi'$

PROOF: THE STATEMENT FOLLOWS IMMEDIATLY FROM THE APPLICATION OF THE BARROW - TORRICELLI THEOREM.

21) EXPECTED VALUE AS STIELTJES INTEGRAL* (2099)

STATEMENT: LET $P: 2^{\Omega} \to \mathbb{R}$ BE A SIMPLE PROBABILITY, $\forall$ RANDOM VARIABLE $f: \Omega \to \mathbb{R}$: $E_P(f) = \int_a^b x \, d\Phi(x)$ WHERE $[a,b]$ IS A CARRIER OF Φ.

PROOF: LET $\text{SUPP}(P) = \{\omega_1, \omega_2 \dots \omega_m\}$ AND SET $f(\omega_k) = x_k$, ASSUMING THE VALUES ARE DISTINCT. NOW LET $x_1 < x_2 \dots < x_m$. SINCE $[a,b]$ IS A CARRIER $\Rightarrow a < x_1$ AND $x_m \leq b$ SO WE HAVE $P(\omega_k) = P(f = x_k)$ AND $E_P(f) = \sum_{k=1}^{m} f(\omega_k) P(\omega_k) = \sum_{k=1}^{m} x_k P(f = x_k)$.

NOW SINCE P IS COUNTABLY ADDITIVE, $\forall k$ WE HAVE THAT $P(f = x_k) = \Phi(x_k) - \lim_{x \to x_k^-} \Phi(x)$, SO $E_P(f) = \sum_{k=1}^{m} x_k \left(\Phi(x_k) - \lim_{x \to x_k^-} \Phi(x) \right)$

$\Phi(x)$ IS INCREASING AND RIGHT CONTINUOUS, SO THE ABOVE IS EQUAL TO THE STIELTJES INTEGRAL, WHERE $a < x_1 < x_m \leq b$, THEREFORE:

$$E_P(f) = \sum_{k=1}^{m} x_k \left(\Phi(x_k) - \lim_{x \to x_k^-} \Phi(x) \right) = \int_a^b x \, d\Phi(x)$$

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DEFINITIONS

SET FUNCTION DEFINITION (2038)

A SET FUNCTION $\mu : 2^{\Omega} \to \mathbb{R}$ IS A RULE THAT ASSOCIATES TO EACH SUBSET A OF Ω ONE, AND ONLY ONE, SCALAR $\mu(A)$.

SET FUNCTION PROPERTIES DEFINITION (2040)

A SET FUNCTION $\mu : 2^{\Omega} \to \mathbb{R}$ IS:

1) GROUNDED IF $\mu(\emptyset) = 0$

2) POSITIVE IF $\mu(A) \geq 0 \quad \forall A$

3) MONOTONE IF $\mu(A) \leq \mu(B)$ WHEN $A \subseteq B$

4) ADDITIVE IF $\mu(A \cup B) = \mu(A) + \mu(B)$ WHEN $A \cap B = \emptyset$

5) NORMALIZED IF $\mu(\Omega) = 1$

MEASURE DEFINITION (2042)

A GROUNDED, POSITIVE, AND ADDITIVE SET FUNCTION IS CALLED A MEASURE. IN THIS CASE WE WRITE: $\mu : 2^{\Omega} \to [0, +\infty)$.

PROBABILITY MEASURE DEFINITION (2046)

A NORMALIZED MEASURE IS CALLED A PROBABILITY MEASURE, IN THIS CASE WE WRITE $P: 2^{\Omega} \to [0,1]$.

SIMPLE PROBABILITY DEFINITION (2053)

A PROBABILITY MEASURE $P: 2^{\Omega} \to [0,1]$ IS SIMPLE IF $\exists$ A FINITE EVENT E SUCH THAT: $P(E) = 1$.

COUNTABLY ADDITIVE PROBABILITY DEFINITION (2059)

A PROBABILITY $P: 2^{\Omega} \to [0,1]$ IS COUNTABLY ADDITIVE IF:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

RANDOM VARIABLE DEFINITION (2066)

A REAL VALUED FUNCTION $f: \Omega \to \mathbb{R}$ DEFINED ON A STATE SPACE IS CALLED A RANDOM VARIABLE.

P-a RANDOM VARIABLES DEFINITION (2069)

TWO RANDOM VARIABLES $f, g: \Omega \to \mathbb{R}$ ARE EQUAL P-ALMOST EVERYWHERE WHEN $P(\omega \in \Omega : f(\omega) = g(\omega)) = 1$ OR MORE COMPACTLY WHEN $P(f = g) = 1$.

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EXPECTED VALUE DEFINITION (2071)

THE EXPECTED VALUE OF A RANDOM VARIABLE $f: \Omega \to \mathbb{R}$ WITH RESPECT TO A SIMPLE PROBABILITY P IS THE QUANTITY: $E_P(f) = \sum_{w \in \text{supp}P} f(w) P(w)$

VARIANCE DEFINITION (2079)

THE VARIANCE OF A RANDOM VARIABLE $f: \Omega \to \mathbb{R}$ WITH RESPECT TO A SIMPLE PROBABILITY P IS THE QUANTITY: $V_P(f) = \sum_{w \in \text{supp}P} E_P(f - E_P(f))^2$

STANDARD DEVIATION DEFINITION

THE STANDARD DEVIATION IS THE SQUARE ROOT OF THE VARIANCE AND IS DENOTED BY $\sigma_P(f) = \sqrt{V_P(f)}$

COVARIANCE DEFINITION (2081)

THE COVARIANCE OF TWO RANDOM VARIABLES $f, g: \Omega \to \mathbb{R}$ WITH RESPECT TO A SIMPLE PROBABILITY P IS THE QUANTITY:

$$Cov_P(f,g) = E_P(f - E_P(f)) \cdot (g - E_P(g))$$

CUMULATIVE DISTRIBUTION FUNCTION DEFINITION (2087)

THE CUMULATIVE DISTRIBUTION FUNCTION OF A RANDOM VARIABLE $f: \Omega \to \mathbb{R}$ IS THE FUNCTION $\Phi(x) = P(f \leq x) \quad \forall x \in \mathbb{R}$.

ESSENTIALLY BOUNDED RANDOM VARIABLE DEFINITION

(2092)

A RANDOM VARIABLE $f: \Omega \to \mathbb{R}$ IS ESSENTIALLY BOUNDED IF $\exists$ SCALARS $m, M \in \mathbb{R}$ SUCH THAT $P(m \leq f \leq M) = 1$.

INTEGRABLE DENSITY FUNCTION DEFINITION

(2095)

A POSITIVE FUNCTION $\phi: \mathbb{R} \to [0, +\infty)$ IS AN INTEGRABLE DENSITY FUNCTION OF Φ IF: $\Phi(x) = \int_{-\infty}^{x} \phi(t)\,dt \quad \forall x \in \mathbb{R}$.

STIELTJES EXPECTED VALUE DEFINITION (2100)

THE EXPECTED VALUE OF A RANDOM VARIABLE $f: \Omega \to \mathbb{R}$ WITH DISTRIBUTION FUNCTION Φ UNDER A PROBABILITY $P: 2^{\Omega} \to [0,1]$ IS THE IMPROPER STIELTJES INTEGRAL: $E_P(f) = \int_{-\infty}^{\infty} x \, d\Phi$, WHEN IT EXISTS.

SUPPORT OF A PROBABILITY DEFINITION

THE SUPPORT OF A PROBABILITY IS THE SET OF ALL OUTCOMES ω THAT HAVE NON-ZERO PROBABILITIES.

FORMALLY: $\mathrm{SUPP}(P) = \{\omega \in \Omega : P(\omega) > 0\}$

CARRIER DEFINITION

AN INTERVAL $[a,b]$ SUCH THAT $\Phi(x) = 0 \quad \forall x \leq a$ AND $\Phi(x) = 1 \quad \forall x \geq b$ IS CALLED A CARRIER OF A DISTRIBUTION

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THEOREMS

ADDITIVITY THEOREM (2044)

LET $\mu : 2^{\Omega} \to [0, +\infty)$ BE A MEASURE. FOR EVERY FINITE COLLECTION $\{A_1, A_2 \dots A_m\}$ OF PAIRWISE DISJOINT SUBSETS OF Ω,

$$\mu\left(\bigcup_{i=1}^{m} A_i\right) = \sum_{i=1}^{m} \mu(A_i)$$

HOLDS.

UNIFORM PROBABILITY THEOREM (2048)

THE UNIFORM PROBABILITY ASSIGNS THE SAME PROBABILITY TO ALL SPACES, SO : $P(\omega) = \frac{1}{|\Omega|} \quad \forall \omega \in \Omega$

DIRAC PROBABILITY THEOREM (2049)

FIX A STATE ω_0 IN ANY STATE SPACE Ω FINITE OR INFINITE. THE SET FUNCTION $P : 2^{\Omega} \to \mathbb{R}$ DEFINED BY:

$$P(A) = \begin{cases} 1 & \text{IF } \omega_0 \in A \\ 0 & \text{IF } \omega_0 \notin A \end{cases}$$

IS DENOTED BY δ_{ω_0}, AND CALLED DIRAC PROBABILITY MEASURE.

POISSON PROBABILITY THEOREM (2050)

TAKE $\Omega = \mathbb{N} = \{0, 1, 2 \dots n\}$. FIX A SCALAR $\lambda > 0$ AND DEFINE THE SCALAR SEQUENCE: $P_n = \frac{\lambda^n}{n!} \cdot e^{-\lambda}$ $\forall n \in \mathbb{N}$.
FOR EACH EVENT $A \subseteq \mathbb{N}$ DEFINE THE SEQUENCE $a_n : \begin{cases} 1 & n \in A \\ 0 & n \notin A \end{cases}$.
NOW DEFINE THE POISSON PROBABILITY $P: 2^{\Omega} \to [0,1]$ BY:

$$P(A) = \sum_{n=0}^{\infty} a_n \cdot P_n \qquad \forall A \subseteq \mathbb{N}$$

GEOMETRIC PROBABILITY THEOREM (2051)

CONSIDER $\Omega = \mathbb{N}$, LET $\{r_n\}$ BE A SEQUENCE OF POSITIVE NUMBERS SUCH THAT THE SERIES $\sum_{n=0}^{\infty} r_n$ CONVERGES WITH SUM R.
DEFINE THE SCALAR SEQUENCE $P_n = \frac{r_n}{R}$.

THE GEOMETRIC PROBABILITY IS DEFINED BY TAKING $r_n = q^n$, $q \in (0,1)$ AS $\sum_{n=0}^{\infty} r_n$ IS THE GEOMETRIC SERIES, IN THIS CASE $P_n = (1-q) \cdot q^n$.
DEFINE THE SET FUNCTION $P: 2^{\Omega} \to [0,1]$ BY $P(A) = \sum_{n=0}^{\infty} a_n P_n$.

PROBABILITY OF ANY ω OUTSIDE THE SUPPORT THEOREM (2055)

LET $P: 2 \to [0,1]$ BE A SIMPLE PROBABILITY MEASURE. IF $\omega \notin E$ THEN $P(\omega) = 0$

PROBABILITY OF A SUPPORT OF A SIMPLE PROBABILITY
(2056)

THE SUPPORT OF A SINGLE PROBABILITY $P: 2^{\Omega} \to [0,1]$ IS A FINITE EVENT WITH PROBABILITY 1, THAT IS $P(\operatorname{supp} P) = 1$.
MOREOVER, $P(A) = 1$ IMPLIES $\operatorname{supp} P \subseteq A$ FOR ALL EVENTS IN A.

PROBABILITY AS A SUM OVER SUPPORT (2057)

LET $P: 2^{\Omega} \to [0,1]$ BE A SIMPLE PROBABILITY MEASURE.
FOR EACH EVENT A : $P(A) = \sum_{\omega \in A \cap \operatorname{supp} P} P(\omega)$

CONVERGENT EVENTS AND COUNTABLE ADDITIVITY
1° (2061)

LET $P: 2^{\Omega} \to [0,1]$ BE A PROBABILITY, THE FOLLOWING STATEMENTS ARE EQUIVALENT :

i) P IS COUNTABLY ADDITIVE
ii) IF $A_m \uparrow A$, THEN $P(A_m) \uparrow P(A)$
iii) IF $A_m \downarrow A$, THEN $P(A_m) \downarrow P(A)$

CONVERGENT EVENTS AND COUNTABLE ADDITIVITY

2° (2063)

LET $P: 2^{\Omega} \to [0,1]$ BE A PROBABILITY, THE FOLLOWING STATEMENTS ARE EQUIVALENT:

i) IF $A_m \uparrow \Omega$, THEN $P(A_m) \uparrow 1$
ii) IF $A_m \uparrow A$, THEN $P(A_m) \uparrow P(A)$
iii) IF $A_m \downarrow A$, THEN $P(A_m) \downarrow P(A)$
iiii) IF $A_m \downarrow \emptyset$, THEN $P(A_m) \downarrow 0$

COUNTABLY ADDITIVE POISSON PROBABILITY THEOREM

(2064)

THE POISSON PROBABILITY IS COUNTABLY ADDITIVE.

NO COUNTABLY ADDITIVE UNIFORM PROBABILITY THEOREM (2065)

THERE IS NO COUNTABLY ADDITIVE UNIFORM PROBABILITY $P: 2^{\mathbb{N}} \to [0,1]$

PERFECT CORRELATION (2085)

LET $P: 2^{\Omega} \to \mathbb{R}$ BE A SIMPLE PROBABILITY, FOR RANDOM VARIABLES $f, g: \Omega \to \mathbb{R}$, NON-CONSTANT P-a IT HOLDS: $|\rho_P(f,g)| = 1$ $\iff$ $\exists$ SCALARS $\alpha \neq 0$ AND β SUCH THAT P-a: $f = \alpha g + \beta$
IN PARTICULAR $\rho_P(f,g) = 1 \iff a > 0$ AND $\rho_P(f,g) = -1 \iff a < 0$

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RIGHT CONTINUITY OF Φ THEOREM (2089)

IF P IS COUNTABLY ADDITIVE, THEN Φ IS RIGHT CONTINUOUS, WITH

$$\lim_{x \to -\infty} \Phi(x) = 0 \quad \text{AND} \quad \lim_{x \to +\infty} \Phi(x) = 1$$

CONTINUITY OF Φ THEOREM (2090)

IF P IS COUNTABLY ADDITIVE, THEN, $\forall x_0 \in \mathbb{R}$:

$$\Phi(x_0) - \lim_{x \to x_0^-} \Phi(x) = P(f = x_0)$$

MOREOVER, Φ IS CONTINUOUS AT $x_0 \in \mathbb{R} \iff P(f = x_0) = 0$

Φ OF THE DIRAC PROBABILITY THEOREM (2091)

LET δ_{w_0} BE THE DIRAC PROBABILITY CENTERED AT SOME STATE $w_0 \in \Omega$. THE DISTRIBUTION FUNCTION OF A RANDOM VARIABLE $f : \Omega \to \mathbb{R}$ IS GIVEN BY :

$$\Phi = \begin{cases} 0 & \text{IF } x < x_0 \\ 1 & \text{IF } x \geq x_0 \end{cases}$$

WHERE $x_0 = f(w_0)$, HERE Φ IS DISCONTINUOUS ONLY AT x_0, WITH JUMP OF SIZE $P(f = x_0) = 1$

UNIFORM DISTRIBUTION FUNCTION AND GAUSSIAN THEOREM (2096)

i) GIVEN ANY TWO SCALARS $a < b$, CONSIDER THE UNIFORM DISTRIBUTION FUNCTION:

$$\Phi(x) = \begin{cases} 0 & \text{IF } x < a \\ \frac{x-a}{b-a} & \text{IF } a \leq x \leq b \\ 1 & \text{IF } x > b \end{cases}$$

THE INTERVAL $[a,b]$ IS THE SMALLEST CARRIER OF Φ. ITS DENSITY FUNCTION, CALLED UNIFORM IS:

$$\phi(x) = \begin{cases} \frac{1}{b-a} & \text{IF } a \leq x \leq b \\ 0 & \text{OTHERWISE} \end{cases}$$

BECAUSE $\int_{-\infty}^{x} \phi(t)\, dt = \int_{a}^{x} \frac{1}{b-a}\, dt = \Phi(x) \quad \forall x \in [a,b]$

AND $\int_{-\infty}^{+\infty} \phi(x)\, dx = 1.$

ii) THE GAUSSIAN DISTRIBUTION FUNCTION IS: $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$

THIS DISTRIBUTION HAS NO CARRIERS, AND ITS DENSITY FUNCTION, CALLED GAUSSIAN, IS $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

BECAUSE $\int_{-\infty}^{+\infty} \phi(t)\, dt = 1$

ESSENTIALLY BOUNDED RANDOM VARIABLE AND E_P THEOREM (2101)

ALL ESSENTIALLY BOUNDED RANDOM VARIABLES HAVE FINITE EXPECTED VALUE.

EXAMPLE 2102

i) FOR THE UNIFORM DENSITY ϕ, IT HOLDS :

$$E_P(f) = \int_{-\infty}^{+\infty} x\,\phi(x)\,dx = \int_a^b x\frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\cdot\frac{(b^2-a^2)}{2} = \frac{a+b}{2}$$

ii) FOR THE GAUSSIAN DENSITY ϕ IT HOLDS:

$$E_P(f) = \int_{-\infty}^{+\infty} x\,\phi(x)\,dx = \int_{-\infty}^{+\infty} x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = \int_0^{+\infty} x\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}}dx + \int_{-\infty}^{0} x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx =$$

$$= \int_0^{+\infty} x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx - \int_{-\infty}^{0}(-x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$= \int_0^{+\infty} x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx - \int_0^{+\infty}(-x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

LINEARITY AND MONOTONICITY OF $E_P(f)$ THEOREM

(2104)

LET $f, g : \Omega \to \mathbb{R}$ BE RANDOM VARIABLES WITH FINITE EXPECTED VALUES, THEN:

i) $E_P(\alpha f + \beta g) = \alpha E_P(f) + \beta E_P(g) \quad \forall \alpha, \beta \in \mathbb{R}$

ii) $E_P(f) \geq E_P(g)$ IF $f \geq g$

iii) $E_P(k) = k \quad \forall k \in \mathbb{R}$

i) AND iii) TOGETHER IMPLY THAT $\forall k \in \mathbb{R}$ $E_P(f+k) = E_P(f) + k$ FOR ALL $f : \Omega \to \mathbb{R}$ WITH FINITE EXPECTED VALUE.

Mathematical Finance

Proofs, Definitions and Theorems

These notes have been compiled by **Luca Penouel** as an additional resource to aid BIEM and BIEF students of the **Mathematics 2 course** (30063).

You will find the proofs, definitions, and theorems listed in the official syllabus for the academic year 2024-2025. The versions of the proofs/definitions/theorems contained herein may differ from those found in the official course textbook; they are adaptations that combine elements from multiple sources to enhance clarity and understanding, while (hopefully) maintaining mathematical correctness.

Please note that this handout IS NOT intended to replace the official course materials. It is provided merely as a supplementary tool to assist in your study and understanding of the course content.

The notes are not in its final form; updates and corrections will be applied as needed. If you notice any errors, kindly contact me privately so that I can make the necessary revisions.

For the most current version of these notes, always access the document via the shared drive link. Avoid downloading the PDF directly, as updates will be posted periodically and downloading may result in using outdated material.

Updated: April 22, 2025

* = IN CLASS
H = HOMEWORK

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PENOUEL

PROOFS

22) CHARACTERIZATION OF DECOMPOSABLE FINANCIAL LAWS* (35)

STATEMENT: LET $f(t)$ BE A CONTINUOUS ACCUMULATION FACTOR, WITH $t \geq 0$. THE ACCUMULATION FACTOR $f(t)$ IS DECOMPOSABLE $\Longleftrightarrow$ $f(t)$ IS A COMPOUND ACCUMULATION LAW.

PROOF:

"ONLY IF": ASSUME THAT $f(t)$ IS A DECOMPOSABLE ACCUMULATION FACTOR SO $\forall t, s \geq 0$, $f(t) = f(s) \cdot f(t-s)$. FROM THE EXPONENTIAL EQUATION THEOREM WE KNOW THAT $f(t) = e^{\delta t}$, WITH $\delta \geq 0$.

"IF": CONSIDER THE COMPOUND ACCUMULATION FACTOR $f(t) = e^{\delta t}$ $\delta \geq 0$. WE HAVE: $f(t) = e^{\delta t} = e^{\delta(t-s+s)} = e^{\delta(t-s)} \cdot e^{\delta s} = f(t-s) \cdot f(s)$ AND $f(t) = e^{\delta t}$, $\delta \geq 0$, IS A DECOMPOSABLE ACCUMULATION FACTOR.

23) FORMULA FOR ORDINARY ANNUITIES* (42)

STATEMENT:

i) THE FINAL VALUE OF AN ORDINARY ANNUITY WITH ANNUAL COMPOUND RATE i IS: $M = R \cdot s_{\overline{m}|i}$

ii) THE PRESENT VALUE OF AN ORDINARY ANNUITY WITH ANNUAL COMPOUND RATE i IS: $A = R \cdot a_{\overline{m}|i}$

WHERE R IS THE INSTALLMENT AND m THEIR NUMBER

PROOF:

i) $$M = \sum_{s=1}^{m} R(1+i)^{m-s} = R(1+i) \sum_{s=1}^{m} \left(\frac{1}{1+i}\right)^{s} = R \cdot \frac{(1+i)^{m}-1}{i} = R \cdot s_{\overline{m}|i}$$

ii) $$A = \sum_{s=1}^{m} R(1+i)^{-s} = R \sum_{s=1}^{m} \left(\frac{1}{1+i}\right)^{s} = R \cdot \frac{1-(1+i)^{-m}}{i} = R \cdot a_{\overline{m}|i}$$

24) FORMULA FOR DUE ANNUITIES^H (46)

STATEMENT:

i) THE FINAL VALUE OF A DUE ANNUITY WITH ANNUAL COMPOUND RATE i IS: $M = R \cdot s_{\ddot{m}|i}$

ii) THE PRESENT VALUE OF A DUE ANNUITY WITH ANNUAL COMPOUND RATE i IS: $A = R \cdot a_{\ddot{m}|i}$

WHERE R IS THE INSTALLMENT AND m THEIR NUMBER

PROOF:

i) $$M = \sum_{s=1}^{m} R(1+i)^{m-(s-1)} = R(1+i)^{m} \sum_{s=1}^{m} \left(\frac{1}{1+i}\right)^{s-1} = R(1+i) \cdot \frac{(1+i)^{m}-1}{i} = R \cdot s_{\ddot{m}|i}$$

ii) $$A = \sum_{s=1}^{m} R(1+i)^{-(s-1)} = R(1+i) \sum_{s=1}^{m} \left(\frac{1}{1+i}\right)^{s} = R(1+i) \cdot \frac{1-(1+i)^{-m}}{i} = R \cdot a_{\ddot{m}|i}$$

25) DURATION AS A DATE OF FINANCIAL IMMUNIZATION* (62)

STATEMENT: GIVEN A FIXED INCOME BOND THAT PAYS:

YEARS	t_1	t_2	...	t_m
INFLOWS	a_1	a_2	...	a_m

IF i) THE YIELD TO MATURITY OF THE BOND AT $t_0 = 0$ IS i

ii) THE RATE i UNDERGOES A VARIATION Δi AT $T \in (0, t_1]$ AND THE NEW RATE $i' = i + \Delta i$ DOES NOT UNDERGO FURTHER VARIATIONS.

THEN: THE BOND INVESTMENT IS FINANCIALLY IMMUNIZED AGAINST THE INTEREST RATE RISK AT THE DATE:

$$z^* = \frac{\sum_{s=1}^{m} t_s \cdot a_s (1+i)^{-t_s}}{\sum_{s=1}^{m} a_s (1+i)^{-t_s}} = D(i)$$

PROOF: IT'S A MINIMUM PROBLEM: $\underset{\text{WITH RESPECT TO } x}{\text{MIN}}\ V(z^*, x)$.

FERMAT: WE WANT i TO BE A (MINIMIZER) STATIONARY POINT: $\left.\frac{\partial V}{\partial x}\right|_{x=i} = 0$

$V(z^*, x) = \sum_{k=1}^{m} a_k (1+x)^{z^*-t_k}$, WE HAVE: $\frac{\partial V}{\partial x} = \sum_{k=1}^{m} a_k (z^*-t_k)(1+x)^{z^*-t_k-1}$

AND $\left.\frac{\partial V}{\partial x}\right|_{x=i} = 0$ IS JUST: $\sum_{k=1}^{m} a_k (z^*-t_k)(1+i)^{z^*-t_k-1} = 0$

WHICH BECOMES $\sum_{k=1}^{m} a_k z^* (1+i)^{z^*-t_k-1} = \sum_{k=1}^{m} a_k t_k (1+i)^{z^*-t_k-1}$

BY GROUPING z^* AND SOLVING FOR IT WE HAVE: $z^* = \frac{\sum_{k=1}^{m} a_k t_k (1+i)^{-t_k}}{\sum_{k=1}^{m} a_k (1+i)^{-t_k}} = D(i)$

FINALLY, WE KNOW THAT i IS A MINIMUM SINCE WE HAVE $\left.\frac{\partial^2 V}{\partial^2 x}\right|_{x=i} > 0$

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26) BOND PRICE VOLATILITY* (64)

STATEMENT: IF THE MARKET PRICE OF A FIXED INCOME BOND IS:
$P(i) = \sum_{s=1}^{m} a_s (1+i)^{-t_s}$, THEN:

i) THE RELATION: $\frac{P'(i)}{P(i)} = -\frac{D(i)}{1+i}$ HOLDS.

ii) A LINEAR APPROXIMATION OF THE VOLATILITY OF THE PRICE IS:

$$\frac{\Delta P(i)}{P(i)} \approx \frac{P'(i)}{P(i)} \Delta i = -\frac{D(i)}{1+i} \Delta i$$

PROOF:

i) SINCE $P'(i) = \sum_{s=1}^{m} -t_s \cdot a_s (1+i)^{-t_s-1} = -(1+i)^{-1} \cdot \sum_{s=1}^{m} t_s \cdot a_s (1+i)^{-t_s}$

WE HAVE:

$$\frac{P'(i)}{P(i)} = -(1+i)^{-1} \cdot \frac{\sum_{s=1}^{m} t_s \cdot a_s (1+i)^{-t_s}}{\sum_{s=1}^{m} a_s (1+i)^{-t_s}} = -\frac{D(i)}{1+i}$$

ii) THE FUNCTION $P(i)$ IS STRICTLY POSITIVE AND DIFFERENTIABLE EVERYWHERE. SO WE CAN CONSIDER THE FIRST ORDER TAYLOR FORMULA AT EVERY POINT $i \in (-1, +\infty)$:

$P(i+\Delta i) - P(i) = P'(i)\Delta i + o(\Delta i)$ AS $\Delta i \to 0$, THEREFORE

$$\frac{\Delta P(i)}{P(i)} = \frac{P(i+\Delta i) - P(i)}{P(i)} = \frac{P'(i)}{P(i)} \Delta i + \frac{o(\Delta i)}{P(i)} = \frac{P'(i)}{P(i)} \Delta i + o(\Delta i)$$

AS $\Delta i \to 0$. BY NEGLECTING $o(\Delta i)$ WE GET THE DESIRED LINEAR APPROXIMATION: $\frac{\Delta P(i)}{P(i)} = \frac{P'(i)}{P(i)} \Delta i = -\frac{D(i)}{1+i} \Delta i$

WE HAVE AN APPROXIMATION OF BOND PRICE VOLATILITY AS A LINEAR FUNCTION OF Δi. THE RATIO: $\boxed{D^*(i) = \frac{D(i)}{1+i}}$ IS CALLED MODIFIED DURATION AND REPRESENTS A MEASURE OF THE VOLATILITY OF THE BOND PRICE: $\frac{\Delta P(i)}{P(i)} \approx -D^*(i)\Delta i$.

THE HIGHER IT IS, THE GREATER THE VOLATILITY.

27) LAW OF ONE PRICE (LOP)* (1207)

STATEMENT: SUPPOSE THE FINANCIAL MARKET $(L, \underline{P})$ SATISFIES THE LOP. THEN THE PRICING RULE $f: W \to \mathbb{R}$ IS LINEAR.

PROOF: FIRST OBSERVE THAT BY THE LOP $v = f \circ R$, THAT IS $v(\underline{x}) = f(R(\underline{x})) \ \forall \ \underline{x} \in \mathbb{R}^m$. LET US PROVE THE LINEARITY OF f. LET $\underline{w}, \underline{w}' \in W$ AND $\alpha, \beta \in \mathbb{R}$. WE WANT TO SHOW THAT $f(\alpha \underline{w} + \beta \underline{w}') = \alpha f(\underline{w}) + \beta f(\underline{w}')$. SINCE $W = im \mathbb{R}$ $\exists$ VECTORS $\underline{\bar{x}}, \underline{\bar{x}}' \in \mathbb{R}^m$ SUCH THAT $R(\underline{\bar{x}}) = \underline{w}$ AND $R(\underline{\bar{x}}') = \underline{w}'$.

BY DEFINITION OF PRICE, OF A REPLICABLE CONTINGENT CLAIM $P_{\underline{w}} = v(\underline{\bar{x}})$ AND $P_{\underline{w}'} = v(\underline{\bar{x}}')$. BY THE LINEARITY OF R AND v WE HAVE:

$$f(\alpha \underline{w} + \beta \underline{w}') = f(\alpha R(\underline{\bar{x}}) + \beta R(\underline{\bar{x}}')) = f(R(\alpha \underline{\bar{x}} + \beta \underline{\bar{x}}') = v(\alpha \underline{\bar{x}} + \beta \underline{\bar{x}}')$$

$$= \alpha v(\underline{\bar{x}}) + \beta v(\underline{\bar{x}}') = \alpha P_{\underline{w}} + \beta P_{\underline{w}'} = \alpha f(\underline{w}) + \beta f(\underline{w}')$$

THEREFORE THE FUNCTION $f: W \to \mathbb{R}$ IS LINEAR ON W.

28) FIRST NO-ARBITRAGE CONDITION AND LOP*

(1210)

STATEMENT: A FINANCIAL MARKET $(L, \underline{P})$ THAT HAS NO ARBITRAGES OF I KIND, SATISFIES THE LOP.

PROOF: BY APPLYING THE FIRST NO-ARBITRAGE CONDITION TO THE PORTFOLIO $-\underline{x}$, WE HAVE:

$-R(\underline{x}) \geq \underline{0} \implies -v(\underline{x}) \geq 0 \quad \forall \underline{x} \in \mathbb{R}^m$ WHICH MEANS:

$R(\underline{x}) \leq 0 \implies v(\underline{x}) \leq 0 \quad \forall \underline{x} \in \mathbb{R}^m$ WHICH IMPLIES:

$R(\underline{x}) = \underline{0} \implies v(\underline{x}) = 0 \quad \forall \underline{x} \in \mathbb{R}^m$

LET $\underline{x}$ AND $\underline{x}'$ BE TWO PORTFOLIOS SUCH THAT $R(\underline{x}) = R(\underline{x}')$. THE LINEARITY OF $\mathbb{R}$ IMPLIES $R(\underline{x} - \underline{x}') = \underline{0}$ AND SO $v(\underline{x}' - \underline{x}) = 0$, i.e. $v(\underline{x}') = v(\underline{x})$ BY THE LINEARITY OF v.

29) LINEAR AND INCREASING PRICING RULE* (1211)

STATEMENT: A COMPLETE FINANCIAL MARKET $(L, \underline{P})$, WITH $\underline{P} \neq \underline{0}$ SATISFIES THE FIRST NO ARBITRAGE CONDITION $\iff$ THE PRICING RULE IS LINEAR AND INCREASING, THAT IS, $\exists$ A UNIQUE VECTOR $\underline{\pi} \in \mathbb{R}^k_+$ SUCH THAT: $\ell(\underline{w}) = \underline{\pi} \cdot \underline{w} \quad \forall \underline{w} \in W$

PROOF:

"IF": LET $R(\underline{x}) \geq 0$. THEN $v(\underline{x}) = \ell(R(\underline{x})) = \underline{\pi} \cdot R(\underline{x}) \geq 0$ BECAUSE $\underline{\pi} \geq \underline{0}$ BY HYPOTHESIS.

"ONLY IF": SINCE THE MARKET IS COMPLETE, WE HAVE $W = im R = \mathbb{R}^k$. BY THE FIRST NO-ARBITRAGE CONDITION AND LOP THEOREM, THE LOP HOLDS AND SO ℓ IS LINEAR. WE NEED TO SHOW THAT ℓ IS INCREASING. SINCE ℓ IS LINEAR, IT IS SUFFICIENT TO SHOW THAT ℓ IS POSITIVE, i.e. THAT $\underline{w} \geq \underline{0} \implies \ell(\underline{w}) \geq 0$.
LET $\underline{w} \in \mathbb{R}^k$ WITH $\underline{w} \geq \underline{0}$ BEING $im R = \mathbb{R}^k$, $\exists\, \underline{\bar{x}} \in \mathbb{R}^m$ SUCH THAT $R(\underline{\bar{x}}) = \underline{w}$.
SO WE HAVE $R(\underline{\bar{x}}) = \underline{w} \geq \underline{0}$ AND SO THE FIRST NO-ARBITRAGE CONDITION IMPLIES $v(\underline{\bar{x}}) \geq 0$
THEREFORE $\ell(\underline{w}) = \ell(R(\underline{\bar{x}})) = v(\underline{\bar{x}}) \geq 0$. WE CONCLUDE THAT THE LINEAR FUNCTION ℓ IS POSITIVE, AND SO INCREASING.
BY THE RIESZ-MARKOV THEOREM, $\exists$ A UNIQUE POSITIVE VECTOR $\underline{\pi} \in \mathbb{R}^k_+$ SUCH THAT: $\ell(\underline{z}) = \underline{\pi} \cdot \underline{z}$, $\forall \underline{z} \in \mathbb{R}^k$.

30) FUNDAMENTAL THEOREM OF FINANCE* (1212)

STATEMENT: A COMPLETE FINANCIAL MARKET $(L, \underline{P})$, WITH $\underline{P} \neq \underline{0}$, SATISFIES THE TWO NO ARBITRAGE CONDITIONS $\Longleftrightarrow$ THE PRICING RULE IS LINEAR AND STRICTLY INCREASING, THAT IS, $\exists$ A UNIQUE VECTOR $\underline{\pi} \in \mathbb{R}^k_{++}$ SUCH THAT: $f(\underline{w}) = \underline{\pi} \cdot \underline{w} \quad \forall \underline{w} \in W$

PROOF:

"IF": LET $R(\underline{x}) > 0$. THEN $v(\underline{x}) = f(R(\underline{x})) = \underline{\pi} \cdot R(\underline{x}) > 0$ BECAUSE $\underline{\pi} \gg \underline{0}$ BY HYPOTHESIS.

"ONLY IF": BY THE PRICING RULE THEOREM f IS LINEAR AND INCREASING. WE NEED TO SHOW THAT f IS STRICTLY INCREASING.
SINCE f IS LINEAR, IT IS SUFFICIENT TO SHOW THAT f IS STRICTLY POSITIVE, i.e. THAT $\underline{w} > \underline{0} \Longrightarrow f(\underline{w}) > 0$.
LET $\underline{w} \in \mathbb{R}^k$ WITH $\underline{w} > 0$. BEING $\operatorname{im} R = \mathbb{R}^k$, $\exists\ \underline{\bar{x}} \in \mathbb{R}^n$ SUCH THAT $R(\underline{\bar{x}}) = \underline{w}$.
SO WE HAVE $R(\underline{\bar{x}}) = \underline{w} > \underline{0}$ AND SO THE SECOND NO-ARBITRAGE CONDITION IMPLIES $v(\underline{\bar{x}}) \geq 0$ BECAUSE OF THE LINEARITY OF v.
THEREFORE $f(\underline{w}) = f(R(\underline{\bar{x}})) = v(\underline{\bar{x}}) > 0$. WE CONCLUDE THAT THE LINEAR FUNCTION f IS STRICTLY POSITIVE, AND SO STRICTLY INCREASING.
BY THE RIESZ-MARKOV THEOREM, $\exists$ A UNIQUE STRONGLY POSITIVE VECTOR $\underline{\pi} \in \mathbb{R}^k_{++}$ SUCH THAT: $f(\underline{z}) = \underline{\pi} \cdot \underline{z}$, $\forall \underline{z} \in \mathbb{R}^k$.

LUCA
PENOUEL

DEFINITIONS

ELEMENTARY FINANCIAL OPERATION DEFINITION (1)

AN EXCHANGE BETWEEN TWO AMOUNTS OF MONEY AVAILABLE ON DIFFERENT DATES IS CALLED AN ELEMENTARY FINANCIAL OPERATION.

ACCUMULATION OPERATION DEFINITION (2)

LET M AND C BE TWO POSITIVE REAL NUMBERS. AN ACCUMULATION OPERATION IS AN EXCHANGE BETWEEN AN AMOUNT OF MONEY AVAILABLE TODAY C (PRINCIPAL OR INVESTED CAPITAL) WITH ANOTHER ONE M (FINAL VALUE OR ACCUMULATED VALUE) AVAILABLE IN THE FUTURE (AT A DATE $t \geq 0$).

DISCOUNT OPERATION DEFINITION (3)

LET S AND A BE TWO POSITIVE REAL NUMBERS. A DISCOUNT OPERATION IS AN EXCHANGE BETWEEN AN AMOUNT OF MONEY S (NOMINAL VALUE) AVAILABLE AT $t \geq 0$ WITH ANOTHER ONE A (PRESENT VALUE OR DISCOUNTED VALUE) AVAILABLE TODAY (AT $t = 0$).

CONJUGATED FINANCIAL FACTORS DEFINITION (7)

LET $f(t)$ AND $e(t)$ BE, RESPECTIVELY, AN ACCUMULATION FACTOR AND A DISCOUNT FACTOR. THE FINANCIAL FACTORS $f(t)$ AND $e(t)$ ARE SAID TO BE CONJUGATED FINANCIAL FACTORS WHEN: $\forall t \geq 0$ $f(t) \cdot e(t) = 1$.

ANNUAL INTEREST RATE DEFINITION (9)

THE INTEREST GENERATED BY € 1 INVESTED FOR THE FIRST YEAR IS INDICATED WITH THE SYMBOL i AND IS CALLED THE ANNUAL INTEREST RATE.

SIMPLE INTEREST DEFINITION

THE INTEREST INDICATED WITH I IS A FUNCTION OF C, t AND i AND IS DEFINED BY THE FORMULA:

$$I = C \cdot i \cdot t \qquad \text{WITH} \quad C, t \geq 0 \quad \text{AND} \quad i > 0$$

SIMPLE DISCOUNT DEFINITION

THE SIMPLE DISCOUNT IS THE CONJUGATE OPERATION OF SIMPLE ACCUMULATION, AND IS DEFINED BY THE DISCOUNT FACTOR:

$$e(t) = \frac{1}{1+it} \qquad \text{FOR} \quad t \geq 0$$

COMPOUND INTEREST DEFINITION

COMPOUND INTEREST IS THE INTEREST CALCULATED ON BOTH THE INITIAL PRINCIPAL AND THE INTEREST ACCUMULATED FROM PREVIOUS PERIODS. THE FINAL VALUE M OF A PRINCIPAL C INVESTED AT TIME 0, AT MATURITY t, IS GIVEN BY:

$$M = C \cdot (1+i)^t \qquad \text{FOR} \quad t \geq 0$$

COMPOUND DISCOUNT DEFINITION

THE COMPOUND DISCOUNT IS THE CONJUGATED OPERATION OF COMPOUND ACCUMULATION, AND IS DEFINED BY THE DISCOUNT FACTOR:

$$e(t) = \frac{1}{(1+i)^t} \quad \text{FOR } t \geq 0$$

CONTINUOUSLY COMPOUNDED INTEREST

CONTINUOUSLY COMPOUNDED INTEREST ASSUMES INTEREST IS CAPITALIZED AT EVERY INSTANT, AND IS DEFINED BY THE FORMULA:

$$M = C \cdot e^{\delta t} \quad \text{FOR } t \geq 0$$

WHERE $\delta \geq 0$ IS CALLED THE FORCE OF INTEREST.

FORCE OF INTEREST DEFINITION

FOR THE FINANCIAL LAW $f(t)$ WE DEFINE THE FORCE OF INTEREST AS :

$$\rho(t) \equiv \lim_{h \to 0} \frac{f(t+h) - f(t)}{f(t)} \cdot \frac{1}{h} \equiv \frac{f'(t)}{f(t)}$$

- FOR SIMPLE INTEREST: $\rho(t) = \frac{1}{1+it}$
- FOR COMPOUND INTEREST: $\rho(t) = \ln(1+i)$

DECOMPOSABILITY DEFINITION (33)

LET $f(t)$ BE AN ACCUMULATION FACTOR. THE ACCUMULATION FACTOR $f(t)$ IS SAID TO BE DECOMPOSABLE WHEN: $f(t) = f(t-s) \cdot f(s)$ $\forall\, s, t$ SUCH THAT $0 \leq s \leq t$.

ANNUITY DEFINITION (36)

THE SEQUENCE OF CASH FLOWS $R_1, R_2 \dots R_m$ WITH THE SAME SIGN AND MATURITIES, RESPECTIVELY, $t_1, t_2 \dots t_m$ IS CALLED AN ANNUITY. IF $R_s \geq 0 \ \forall s$ THEN THE CASH FLOWS ARE ALL INFLOWS, MEANWHILE IF $R_s \leq 0 \ \forall s$, THEN THE CASH FLOWS ARE ALL OUTFLOWS.

ORDINARY ANNUITY DEFINITION (38)

LET s BE A NATURAL NUMBER. THE SEQUENCE OF CASH FLOWS $R_1, R_2 \dots R_m$ WITH THE SAME SIGN AND MATURITIES, RESPECTIVELY, $1, 2, \dots m$ IS CALLED AN ORDINARY ANNUITY. IF $R_s \geq 0 \ \forall s$, THEN THE CASH FLOWS ARE ALL INFLOWS, MEANWHILE IF $R_s \leq 0 \ \forall s$, THEN THE CASH FLOWS ARE ALL OUTFLOWS.

DUE ANNUITY DEFINITION (44)

LET s BE A NATURAL NUMBER. THE SEQUENCE OF CASH FLOWS $R_1, R_2 \dots R_m$ WITH THE SAME SIGN AND MATURITIES RESPECTIVELY, $0, 1, \dots m-1$ IS CALLED A DUE ANNUITY. IF $R_s \geq 0 \ \forall s$, THEN THE CASH FLOWS ARE ALL INFLOWS, MEANWHILE IF $R_s \leq 0 \ \forall s$, THEN THE CASH FLOWS ARE ALL OUTFLOWS.

ORDINARY PERPETUITY DEFINITION (47)

A FINANCIAL OPERATION DESCRIBED BY THE FOLLOWING TABLE

YEARS	$t_0 = 0$	1	2	3	...	$m-1$	m	...
CASH FLOWS		R	R	R	...	R	R	...

IS CALLED AN ORDINARY PERPETUITY (WHERE THE CASH FLOWS ARE INFINITELY MANY, AND FOR SIMPLICITY WE ASSUME THAT THEY ALL HAVE THE SAME VALUE).

DUE PERPETUITY DEFINITION (50)

A FINANCIAL OPERATION DESCRIBED BY THE FOLLOWING TABLE

YEARS	$t_0=0$	1	2	3	...	m-1	m	...
CASH FLOWS	R	R	R	R	...	R	R	...

IS CALLED A DUE PERPETUITY (WHERE THE CASH FLOWS ARE INFINITELY MANY, AND FOR SIMPLICITY WE ASSUME THAT THEY ALL HAVE THE SAME VALUE).

DCF DEFINITION

YEARS	$t_0=0$	t_1	t_2	...	$t_m=T$
CASH FLOW	a_0	a_1	a_2	...	a_m

CONSIDER THE FINANCIAL OPERATION:

THE FUNCTION $g:(-1,+\infty)\rightarrow\mathbb{R}$ DEFINED BY $G(x)=\sum_{s=0}^{m}\frac{a_s}{(1+x)^{t_s}}$

IS THE DISCOUNTED CASH FLOW ASSOCIATED TO THE FINANCIAL OPERATION. IT DESCRIBES THE PRESENT VALUE OF THE CASH FLOW, WHERE THE COMPOUND ANNUAL INTEREST RATE IS TAKEN AS A VARIABLE.

NPV DEFINITION

YEARS	$t_0=0$	t_1	t_2	...	$t_m=T$
CASH FLOW	a_0	a_1	a_2	...	a_m

CONSIDER THE FINANCIAL OPERATION:

AND ITS DISCOUNTED CASH FLOW $G(x)$.

THE AMOUNT $G(i)$ IS CALLED NET PRESENT VALUE OF THE FINANCIAL OPERATION AT THE RATE i.

$$G(i)=\sum_{s=0}^{m}\frac{a_s}{(1+i)^{t_s}}$$

INVESTMENT DEFINITION

A FINANCIAL OPERATION, WITH MATURITY T, DESCRIBED BY THE FOLLOWING TABLE:

YEARS	$t_0 = 0$	t_1	t_2	t_3	$\cdots$	t_{m-1}	$t_m = T$
CASH FLOWS	a_0	a_1	a_2	a_3	$\cdots$	a_{m-1}	a_m

IS CALLED AN INVESTMENT IF: $a_0 < 0$ AND $a_s \geq 0$, $\forall s = 1, 2, \dots, m$ WITH AT LEAST AN $a_s > 0$

LOAN DEFINITION

A FINANCIAL OPERATION, WITH MATURITY T, DESCRIBED BY THE FOLLOWING TABLE:

YEARS	$t_0 = 0$	t_1	t_2	t_3	$\cdots$	t_{m-1}	$t_m = T$
CASH FLOWS	a_0	a_1	a_2	a_3	$\cdots$	a_{m-1}	a_m

IS CALLED A LOAN IF: $a_0 > 0$ AND $a_s \leq 0$, $\forall s = 1, 2, \dots, m$ WITH AT LEAST AN $a_s < 0$

INTERNAL RATES DEFINITION

THE INTERNAL RATE OF A FINANCIAL OPERATION IS ANY SOLUTION OF THE EQUATION $G(x) = 0$.

- IF IT IS A TYPICAL INVESTMENT: IRR (INTERNAL RATE OF RETURN)
- IF IT IS A TYPICAL LOAN: EC (EFFECTIVE COST)

IRR AND EFFECTIVE COST DEFINITION (53)

THE INTERNAL RATE OF AN INVESTMENT IS CALLED THE INTERNAL RATE OF RETURN (OR IRR), WHILE THE INTERNAL RATE OF A LOAN IS CALLED THE EFFECTIVE COST OF THE LOAN.

FIXED-INCOME BOND DEFINITION

A FIXED INCOME BOND IS A SECURITY THAT CONTRACTUALLY GUARANTEES FUTURE PAYMENTS BOTH IN TERMS OF THE AMOUNTS AND OF THE MATURITY DATES.

YIELD TO MATURITY OF A ZCB DEFINITION (55)

THE YIELD TO MATURITY, OR GROSS COUPON YIELD, OF A ZCB IS THE GROSS COMPOUND RATE OF RETURN $i = i(0,T)$ THAT CHARACTERIZES THE INVESTMENT FROM 0 UNTIL THE MATURITY $T \geq 0$ OF THE ZCB.

$$i(0,T) = \left(\frac{N}{P_0}\right)^{\frac{1}{T}} - 1$$

YIELD TO MATURITY OF A BOND WITH COUPONS DEFINITION (57)

THE YIELD TO MATURITY (YTM) OF THE BOND DESCRIBED BY THE CASH FLOWS:

YEARS	$t_0 = 0$	t_1	t_2	...	$t_m = T$
CASH FLOWS	$-P_0$	c_1	c_2	...	$c_m + R$

IS THE COMPOUND INTEREST RATE x^* WHICH SOLVES THE EQUATION:

$$G(x) = -P_0 + \sum_{s=1}^{m} c_s (1+x)^{-t_s} + R(1+x)^{-t_m} = 0$$

DURATION DEFINITION (59)

GIVEN THE CASH FLOW: (A):

YEARS	t_1	t_2	...	t_m
INFLOWS	a_1	a_2	...	a_m

WITH $a_s > 0$, $s = 1, \dots, m$

THE (MACAULAY) DURATION OF (A), CALCULATED AT THE (ANNUAL COMPOUND INTEREST) RATE $i > -1$ IS THE QUANTITY:

$$D(i) = \frac{\sum_{s=1}^{m} t_s \cdot a_s (1+i)^{-t_s}}{\sum_{s=1}^{m} a_s (1+i)^{-t_s}}$$

FINANCIAL IMMUNIZATION DEFINITION (61)

AN INVESTMENT AT RATE i IS FINANCIALLY (GLOBALLY) IMMUNIZED AT YEAR z^* AGAINST THE INTEREST RATE RISK IF:

$$V(z^*, i + \Delta i) \geq V(z^*, i) \quad \forall \ \Delta i$$

AND THE LIFETIME z^* IS CALLED THE DATE OF IMMUNIZATION.

VOLATILITY DEFINITION

THE VOLATILITY OF THE PRICE OF A BOND IS THE SENSITIVITY OF THE PRICE TO A CHANGE IN YIELD RATE: $\frac{\Delta P(i)}{P(i)} = \frac{P(i+\Delta i) - P(i)}{P(i)}$ EXPRESSED IN PERCENTAGE.

MODIFIED DURATION DEFINITION

WE CALL MODIFIED DURATION OF A SECURITY AT THE MARKET RATE i, THE VALUE: $D^*(i) = \frac{D(i)}{1+i}$

PORTFOLIO DEFINITION

A PORTFOLIO IS A VECTOR $\underline{x} = [x_1, x_2 \dots x_m] \in \mathbb{R}^m$ THAT REPRESENTS THE QUANTITIES OF m PRIMARY ASSETS TRADED ON THE MARKET.

PAYOFF DEFINITION

THE PAYOFF OF AN ASSET IS ITS MONETARY OUTCOME IN EACH POSSIBLE STATE OF THE WORLD. FOR A PRIMARY ASSET j, THE PAYOFF IS REPRESENTED BY A VECTOR $\underline{y}_j = [y_{1j}, y_{2j} \dots y_{kj}] \in \mathbb{R}^k$. WHERE y_{ij} IS THE PAYOFF IF STATE S_i OCCURS.

CONTINGENT CLAIM DEFINITION

WE CALL CONTINGENT CLAIM ANY STATE CONTINGENT PAYOFF $\underline{w} \in \mathbb{R}^k$

REPLICABLE CONTINGENT CLAIM DEFINITION

A CLAIM $\underline{w} \in \mathbb{R}^k$ IS REPLICABLE IN THE MARKET IF $\exists$ A PORTFOLIO SUCH THAT: $\underline{w} = \sum_{j=1}^{m} \underline{x_j} \, \underline{y_j}$

COMPLETE FINANCIAL MARKET DEFINITION

THE MARKET W IS COMPLETE IF $W \equiv \mathbb{R}^k$, THAT IS, IF ALL CONTINGENT CLAIMS ARE REPLICABLE.

INCOMPLETE FINANCIAL MARKET DEFINITION

THE MARKET W IS INCOMPLETE IF $W \subset \mathbb{R}^k$, THAT IS, IF NOT ALL CONTINGENT CLAIMS ARE REPLICABLE.

ARROW CONTINGENT CLAIM DEFINITION

AN ARROW (OR PURE) CONTINGENT CLAIM $\underline{e}^i \in \mathbb{R}^k$ IS A VECTOR THAT PAYS ONE UNIT IF STATE S_i OCCURS AND 0 IN ALL OTHER STATES.

PAYOFF OPERATOR DEFINITION

THE PAYOFF OPERATOR IS THE FUNCTION ASSOCIATING TO EACH PORTFOLIO $\underline{x}$ THE CLAIM $\underline{w}$ THAT IT INDUCES: $\mathbb{R}: \mathbb{R}^m \rightarrow \mathbb{R}^k$

$$\underline{x} \rightarrow \mathbb{R}(\underline{x}) = \underline{w} = \sum_{j=1}^{m} \underline{x}\,\underline{y}$$

PAYOFF MATRIX DEFINITION

THE PAYOFF MATRIX $\underset{k \times m}{Y} = (y_{ij}) = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1m} \\ y_{21} & y_{22} & \cdots & y_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ y_{k1} & y_{k2} & \cdots & y_{km} \end{bmatrix}$ IS A k BY m MATRIX THAT REPRESENTS THE PAYOFFS OF m PRIMARY ASSETS ACROSS k STATES.

MARKET VALUE DEFINITION

THE MARKET VALUE OF THE PORTFOLIO $\underline{x}$ IS THE FUNCTION ASSOCIATING TO EACH PORTFOLIO ITS PRICE TODAY: $v: \mathbb{R}^m \rightarrow \mathbb{R}$

$$\underline{x} \rightarrow v(\underline{x}) = \underline{P} \cdot \underline{x} = \sum_{j=1}^{m} x_j P_j$$

LUCA
PENOUEL

LAW OF ONE PRICE DEFINITION (1204)

THE FINANCIAL MARKET (L, P) SATISFIES THE LAW OF ONE PRICE (LOP) IF, $\forall$ PORTFOLIO $\underline{x}, \underline{x}' \in \mathbb{R}^m$, $R(\underline{x}) = R(\underline{x}') \Rightarrow v(\underline{x}) = v(\underline{x}')$.

PRICE OF A REPLICABLE CONTINGENT CLAIM DEFINITION (1205)

THE PRICE $P_{\underline{\omega}}$ OF A REPLICABLE CONTINGENT CLAIM $\underline{\omega} \in W$ IS THE VALUE OF A REPLICATING PORTFOLIO $\underline{x} \in \mathbb{R}^{-1}(\underline{\omega})$, THAT IS, $P_{\underline{\omega}} = v(\underline{x})$ WHERE $\underline{\omega} = R(\underline{x})$.

PRICING KERNEL DEFINITION

SUPPOSE THE FINANCIAL MARKET $\{L, \underline{P}\}$ SATISFIES THE LOP, THEN $\exists$ A UNIQUE VECTOR $\underline{\pi} \in W$ SUCH THAT: $f(\underline{\omega}) = \underline{\pi} \cdot \underline{\omega} \quad \forall\, \underline{\omega} \in W \subseteq \mathbb{R}^k$

SUCH VECTOR π IS CALLED THE PRICING KERNEL OF THE CLAIMS.

ARBITRAGE DEFINITIONS

ARBITRAGE OF I KIND: $\begin{cases} \underline{Y} \cdot \underline{x} = R(\underline{x}) \geq 0 \\ \underline{P} \cdot \underline{x} = v(\underline{x}) < 0 \end{cases}$

ARBITRAGE OF II KIND: $\begin{cases} \underline{Y} \cdot \underline{x} = R(\underline{x}) > 0 \\ \underline{P} \cdot \underline{x} = v(\underline{x}) \leq 0 \end{cases}$

NO ARBITRAGE CONDITIONS DEFINITION

A MARKET SATISFIES THE FIRST NO-ARBITRAGE CONDITION IF:

I: $R(\underline{x}) \geq \underline{0} \implies \upsilon(\underline{x}) \geq 0 \quad \forall\, \underline{x} \in \mathbb{R}^n$

A MARKET SATISFIES THE SECOND NO-ARBITRAGE CONDITION IF:

I: $R(\underline{x}) > \underline{0} \implies \upsilon(\underline{x}) > 0 \quad \forall\, \underline{x} \in \mathbb{R}^n$

CAPITALIZATION AXIOMS

AXIOM 1: DEPENDENCE OF M

THE FINAL VALUE M DEPENDS ON C AND t, THEREFORE WE CAN WRITE: $M = M(c, t)$

AXIOM 2: ADDITIVITY WITH RESPECT TO THE PRINCIPAL

LET C_1, C_2 BE TWO CAPITALS, WE HAVE: $\forall\, C_1, C_2 \geq 0$,
$M(C_1 + C_2, t) = M(C_1, t) + M(C_2, t)$

AXIOM 3: INCREASING MONOTONICITY WITH RESPECT TO TIME

LET t_1, t_2 BE TWO LIFETIMES, WE HAVE: $\forall\, t_1, t_2 \geq 0$
$t_1 < t_2 \implies M(c, t_1) \leq M(c, t_2)$

AXIOM 4: ENGAGEMENT RULE

INITIALS EXTRA COSTS ARE EXCLUDED, SO IF THE LIFETIME IS
$t = 0 \implies M(c, 0) = C$

THEOREMS

CAUCHY THEOREM (11)

LET f BE A CONTINUOUS FUNCTION AT LEAST ONE POINT $x_0 \in \mathbb{R}$, THEN:

$$\forall x, y \in \mathbb{R} \qquad f(x+y) = f(x) + f(y) \iff f(x) = ax, \quad a \in \mathbb{R}$$

EXPONENTIAL CAUCHY EQUATION THEOREM (12)

LET f BE A STRICTLY POSITIVE FUNCTION, CONTINUOUS AT LEAST AT ONE POINT $x_0 \in \mathbb{R}$, THEN:

$$\forall x, y \in \mathbb{R} \qquad f(x+y) = f(x) \cdot f(y) \iff f(x) = e^{mx}, \quad m \in \mathbb{R}$$

AXIOM SATISFYING FUNCTIONS THEOREM (14)

LET $C \geq 0$, $t \geq 0$ BE, RESPECTIVELY, A CAPITAL AND LIFETIME ALL FUNCTIONS $M(C,t)$ WHICH SATISFY $A_1 - A_4$ AXIOMS ARE $M(C,t) = C\,f(t)$ WITH THE FOLLOWING PROPERTIES:

1) $f(0) = 1$
2) f IS INCREASING

EQUIVALENT RATES IN COMPOUND INTEREST

TWO INTEREST RATES ARE CALLED EQUIVALENT IF IN THE SAME LIFETIME THEY PRODUCE THE SAME FINAL VALUE STARTING FROM THE SAME PRINCIPAL.

- FOR SIMPLE INTEREST: $i \equiv m \cdot i_m$
- FOR COMPOUND INTEREST: $1+i \equiv (1+i_m)^m$

REMARK ON ORDINARY PERPETUITY (48)

IN COMPOUND INTEREST, THE PRESENT VALUE A OF AN ORDINARY PERPETUITY IS DEFINED BY:

$$A = \frac{R}{1+i} + \frac{R}{(1+i)^2} + \frac{R}{(1+i)^3} + \dots + \frac{R}{(1+i)^{m-1}} + \frac{R}{(1+i)^m} + \dots$$, THEREFORE

$$A = \sum_{s=1}^{+\infty} R(1+i)^{-s} = R \lim_{m \to +\infty} \sum_{s=1}^{m} (1+i)^{-s} = R \lim_{m \to +\infty} \sum_{s=1}^{m} \left(\frac{1}{1+i}\right)^{s}$$

$$= R \cdot \lim_{m \to +\infty} \frac{1-(1+i)^{-m}}{i} = \frac{R}{i} = R \cdot a_{\overline{\infty}|i}$$

N.B.: $(1+i)^{-m} \to 0$
AS $m \to +\infty$

REMARK ON DUE PERPETUITY (51)

IN COMPOUND INTEREST, THE PRESENT VALUE A OF A DUE PERPETUITY IS DEFINED BY:

$$A = R + \frac{R}{1+i} + \frac{R}{(1+i)^2} + \frac{R}{(1+i)^3} + \dots + \frac{R}{(1+i)^{m-1}} + \frac{R}{(1+i)^m} + \dots$$, THEREFORE

$$A = \sum_{s=0}^{+\infty} R(1+i)^{-s} = R \lim_{m \to +\infty} \sum_{s=0}^{m} (1+i)^{-s} = R \cdot \lim_{m \to +\infty} \left[1 + \sum_{s=1}^{m} (1+i)^{-s}\right]$$

$$= R + \frac{R}{i} = R \cdot \frac{1+i}{i} = R \cdot \ddot{a}_{\overline{\infty}|i}$$

RIESZ - MARKOV THEOREM (673)

A FUNCTION $f : \mathbb{R}^n \to \mathbb{R}$ IS LINEAR AND INCREASING $\Longleftrightarrow$ $\exists$ A UNIQUE POSITIVE VECTOR $\underline{\chi} \in \mathbb{R}^n_+$ SUCH THAT: $f(\underline{x}) = \underline{\chi} \cdot \underline{x} \quad \forall \underline{x} \in \mathbb{R}^n$, AND IN PARTICULAR

i) $\underline{\chi} > \underline{0} \Longleftrightarrow f$ IS STRONGLY INCREASING

ii) $\underline{\chi} \gg \underline{0} \Longleftrightarrow f$ IS STRICTLY INCREASING

REPRESENTATION RESULT FOR THE PRICING RULE THEOREM (1208)

SUPPOSE THE FINANCIAL MARKET (L, p) SATISFIES THE LOP. THEN $\exists$ A UNIQUE VECTOR $\underline{\pi} \in W$ SUCH THAT:

$$f(\omega) = \underline{\pi} \cdot \omega \qquad \forall \omega \in W$$

FOR DOUBTS OR SUGGESTIONS ON THE HANDOUTS

LUCA PENOUEL

Representative of the Department of Decision Sciences

luca.penouel@studbocconi.it
@luca.penouel
+39 3338998638

FOR INFO ON THE TEACHING DIVISION

VITTORIA NASONTE

vittoria.nasonte@studbocconi.it
@_vittorian_
+39 3274441476

ELENA CACIOLI

elena.cacioli@studbocconi.it
@elenacaciolii_
+39 3928931605

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