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BLAB

HANDOUTS

**MATHEMATICS
MODULE II
(APPLIED)
-DEFINITIONS-**

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This handout is written by students with no intention of replacing university materials.

It is a useful tool for studying the subject, but does not guarantee preparation as exhaustive and complete as the material recommended by the University.



Mathematics Module II (Applied)

Definitions

Michele Rossini - BIEM15 - AY: 2023-2024

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Probability

Event

The event E is every subset of a sample space $\Omega : E \subseteq \Omega$

Power set

$\forall \Omega$ the power set is the set made of all possible events:

$$2^\Omega = \cup_{E \subseteq \Omega} E = \{every\ E | E \subseteq \Omega\}$$

Set function

A set function is any function

$$\mu : 2^\Omega \rightarrow R$$

That associates to any event $E \subseteq \Omega$ 1 and only 1 scalar value $\mu(E)$

Measure

A set function

$$\mu : 2^\Omega \rightarrow R$$

That is **grounded, positive, additive** is called a (positive) measure $\mu(E)$ of an event E

The **properties**:

- **Grounded:** if $\mu(\emptyset) = 0$
- **Positive:** if $\mu(A) \geq 0$
- **Monotone:** if $A \subseteq B$ then $\mu(A) \leq \mu(B)$
- **Additive:** if $A \cap B = \emptyset$ then $\mu(A \cup B) = \mu(A) + \mu(B)$

Finite Additivity

A probability $P : 2^\Omega \rightarrow [0,1]$ is finite additive if for any finite collection of pairwise disjoint sets $\{A_i\}_{i=1}^n$ (i.e. $A_i \cap A_j = \emptyset \forall i \neq j$) then

$$\mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$$

Probability Measure

A normalized measure is called probability measure:

$$P : 2^\Omega \rightarrow [0,1]$$

The **properties**:

- **Grounded:** if $P(\emptyset) = 0$
- **Positive:** if $P(A) \geq 0$
- **Monotone:** if $A \subseteq B$ then $P(A) \leq P(B)$
- **Additive:** if $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$
- **Normalized:** if $P(\Omega) = 1$

Simple Probability

A probability measure $P : 2^\Omega \rightarrow [0,1]$ is simple if \exists :

- A finite event E (i.e. an event made by a finite number of outcomes)
- Such that $P(E) = 1$

Support of a Probability

Consider a probability measure $P : 2^\Omega \rightarrow [0,1]$ we call support of P the set

$$\text{supp}P = \{\omega \in \Omega \mid P(\omega) > 0\}$$

Countable additivity

A probability $P : 2^\Omega \rightarrow [0,1]$ is countably additive if for any countable collection of pairwise disjoint sets $\{A_i\}_{i=1}^{+\infty}$ (i.e. $A_i \cap A_j = \emptyset \forall i \neq j$) then

$$P(\cup_{i=1}^{+\infty} A_i) = \sum_{i=1}^{+\infty} P(A_i)$$

About countable additivity (Th)

Let $P : 2^\Omega \rightarrow [0,1]$ be a probability. Then these statements are equivalent:

- P is countably additive
- If $A_n \uparrow A \implies P(A_n) \uparrow P(A)$
- $A_n \downarrow A \implies P(A_n) \downarrow P(A)$

Th

Let $P : 2^\Omega \rightarrow [0,1]$ be a probability. The following statements are equivalent:

- $A_n \uparrow \Omega \implies P(A_n) \uparrow 1$
- $A_n \uparrow A \implies P(A_n) \uparrow P(A)$
- $A_n \downarrow A \implies P(A_n) \downarrow P(A)$
- $A_n \downarrow \emptyset \implies P(A_n) \downarrow 0$

Th

The Poisson probability is countable additive

Random variable

A real valued function

$$f : \Omega \rightarrow R$$

$$\omega \rightarrow f(\omega)$$

With domain a state space is called a random variable.

Expected Value

The expected value of the random variable $f : \Omega \rightarrow R$ with respect to the simple probability P is

$$E_p(f) = \sum_{\omega \in \text{Supp}P} f(\omega)P(\omega)$$

Variance & Standard Deviation

The variance of the random variable $f : \Omega \rightarrow R$ with respect to the simple probability P is

$$V_p(f) = \sum_{\omega \in \text{Supp}P} (f(\omega) - E_p(f))^2 P(\omega)$$

Its standard deviation is

$$\sigma_p(f) = \sqrt{V_p(f)}$$

Properties of the Variance and Standard Deviation:

$$V_p(f) \geq 0 \qquad \sigma_p(f) \geq 0$$

$$V_p(f) = E_p(f^2) - (E_p(f))^2$$

$$V_p(\alpha f + \beta) = \alpha^2 V_p(f)$$

$$\sigma_p(\alpha f + \beta) = |\alpha| \sigma_p(f)$$

Random variables P-equal a.e.

2 random variables (defined on the sample space)

$$f, g : \Omega \rightarrow R$$

$$\omega \rightarrow f(\omega) \text{ and } g(\omega)$$

are P-equal almost everywhere (P-a.e.) if

$$P(\omega \in \Omega : f(\omega) = g(\omega)) = 1$$

Covariance

The covariance of 2 random variables

$$f, g : \Omega \rightarrow R$$

$$\omega \rightarrow f(\omega) \text{ and } g(\omega)$$

With respect to the simple probability P is:

$$COV_p(f, g) = \sum_{\omega \in \text{Supp}P} (f(\omega) - E_p(f))(g(\omega) - E_p(g))P(\omega)$$

Properties of the Covariance:

$$COV_p(F, f) = V_p(f) = \sigma^2(f)$$

$$COV_p(g, f) = E_p((g - E_p(g))(f - E_p(f))) = COV_p(f, g)$$

$$COV_p(\alpha f, \alpha f) = \alpha^2 V_p(f)$$

$$COV_p(\alpha f, \gamma g) = \alpha \gamma COV_p(f, g)$$

$$COV_p(f, \delta) = 0$$

$$-\sigma_p(f)\sigma_p(g) \leq COV_p(f, g) \leq \sigma_p(f)\sigma_p(g)$$

Pearson coefficient

$$\rho = \frac{COV_p(f, g)}{\sigma_p(f)\sigma_p(g)}$$

Bounded:
$$-1 \leq \frac{COV_p(f, g)}{\sigma_p(f)\sigma_p(g)} \leq 1$$

Properties of the correlation coefficient (Th)

Let $P : 2^\Omega \rightarrow [0,1]$ be a simple probability.

For all random variables $f, g : \Omega \rightarrow R$ such that $f(\omega) \neq g(\omega)$ for some $\omega \in \text{Supp}P$ and f and g are not constant on the support, then

$$|\rho_p(f, g)| = 1 \iff \exists 0 \neq \alpha, \beta \mid f = \alpha g + \beta$$

In particular:

- $\rho_p(f, g) = 1 \iff \alpha > 0$
- $\rho_p(f, g) = -1 \iff \alpha < 0$

Cumulative distribution function

The cumulative distribution function of the random variable $f : \Omega \rightarrow R$ under the probability $P : 2^\Omega \rightarrow [0,1]$, is the function

$$\Phi : R \rightarrow [0,1]$$

$$x \rightarrow \Phi(x) = P(f \leq x)$$

The (cumulative) distribution function is right continuous with limits 0 and 1 (Th)

If P is countably additive, then

- Φ is continuous from the right
- $\lim_{x \rightarrow -\infty} \Phi(x) = 0$ and $\lim_{x \rightarrow +\infty} \Phi(x) = 1$

Steps or not for the (cumulative) distribution function

If P is countably additive, then at each x_0

- $\Phi(x) - \lim_{x \rightarrow x_0^-} \Phi(x) \equiv P(f = x_0)$

Every step of the cumulative distribution function corresponds to a concentrated probability

- Φ is continuous at $x_0 \iff P(f = x_0) = 0$

The (cumulative) distribution function has no step at $x_0 \iff$ the corresponding point wise probability is 0, i.e. there is no concentrated probability.

Essentially bounded random variable

A random variable $f : \Omega \rightarrow R$ is essentially bounded if \exists 2 scalars m and M such that:

$$P(\omega : m \leq f(\omega) \leq M) = 1$$

Carrier

An interval $[a, b]$ is a carrier for a cumulative distribution function Φ if

$$\Phi(x) = 0 \forall x \leq a \quad \text{and} \quad \Phi(x) = 1 \forall x \geq b$$

Simple random variable

A random variable is simple if it takes only a finite number of distinct values

Simple density function φ of Φ

A positive function $\varphi : R \rightarrow [0,1]$ non zero only at finitely many points $\{y_1, \dots, y_n\}$ is a simple density function of Φ if

$$\Phi(x) = \sum_{i: y_i \leq x} \varphi(y_i) \quad \forall x \in R$$

Integrable density function φ of Φ

A positive function $\varphi : R \rightarrow [0,1]$ is an integrable density function φ of Φ if

$$\Phi(x) = \int_{-\infty}^x \varphi(t) dt \quad \forall x \in R$$

Gauss Cumulative distribution function

Recall that

$$\int_{-\infty}^{+\infty} e^{-t^2/2} dt = \sqrt{2\pi}$$

The gauss cumulative distribution function is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2/2} dt$$

Corresponding to the continuous (and integrable) Gauss density function

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

This distribution has no carrier.

Expected Value

The expected value of the random variable $f : \Omega \rightarrow R$ with (cumulative) distribution function Φ under a probability $P : 2^\Omega \rightarrow [0,1]$ is the improper Stieltjes' integral

$$E_p(f) = \int_{-\infty}^{+\infty} x d\Phi(x)$$

When it \exists

Th

All essentially bounded random variables have finite expected value

Cavalieri formula (Th)

If the random variable $f : \Omega \rightarrow R$ is essentially bounded, it holds

$$E_p(f) = \int_0^{+\infty} (1 - \Phi(x)) dx - \int_{-\infty}^0 \Phi(x) dx$$

Financial Calculus

Elementary financial operation

An exchange between 2 amounts of money available at different maturities is called an elementary financial operation.

Accumulation

Let M, C be positive real numbers. An accumulation operation is an exchange between the amount C (principal or invested capital) available today with the amount M (final value or accumulated value) available in the future (at time $t \geq 0$)

Discount

Let S, A be positive real numbers. A discount operation is an exchange between the amount S (nominal value) available in the future (at time $t \geq 0$) with the amount A (present value or discounted value) available now (at $t = 0$).

Conjugated financial factors

Let $f(t)$ and $\varphi(t)$ be, respectively, an accumulation and a discount (financial) factors. They are conjugated financial factors if

$$f(t) \cdot \varphi(t) = 1$$

$$\forall t \geq 0$$

Cauchy functional equation (Th)

Let $f : R \rightarrow R$ be a continuous function at (at least) one point x_0 . Then

$$f(x_1 + x_2) = f(x_1) + f(x_2) \quad \forall x_1, x_2 \in R \iff f(x) = a \cdot x \quad \text{with } a \in R$$

Th

Let $f : R \rightarrow R$ be a positive and continuous function at (at least) one point x_0 . Then

$$f(x_1 + x_2) = f(x_1)f(x_2) \quad \forall x_1, x_2 \in R \iff f(x) = e^{m \cdot x} \quad \text{with } m \in R$$

Capitalization axioms

1. Dependence of the amount available

The final value M depends on C and t only, therefore we write:

$$M = M(C, t)$$

2. Additivity with respect to the principal

Let $C_1 \geq 0$ and $C_2 \geq 0$ be 2 principals, then $\forall t \geq 0$ we have

$$M(C_1 + C_2, t) = M(C_1, t) + M(C_2, t)$$

3. Increasing monotonicity with respect to time

Let t_1, t_2 be 2 lifetimes such that $0 \leq t_1 \leq t_2$ then the capital does not lose value over time

$$M(C, t_1) \leq M(C, t_2)$$

4. Engagement rule

Initial extra costs are excluded

$$M(C, 0) = C$$

Th

Let $C \geq 0$ and $t \geq 0$ be a principal and a lifetime.

All the functions $M = (C, t)$ which satisfy the axioms are of the pattern

$$M(C, t) = f(t) \cdot C$$

And the function f will be the corresponding financial accumulation factor). Moreover

- $f(0) = 1$
- f is increasing.

Simple Interest Accumulation

The simple interest accumulation factor is $f(t) = 1 + it$

Compound Interest Accumulation

The compound interest accumulation factor is $f(t) = (1 + i)^t$

Equivalent Rates

2 interest rates are called equivalent if in the same lifetime they produce the same final value starting from the same principal.

For simple interest: $i \equiv m \cdot i_m$

For compound interest: $1 + i \equiv (1 + i_m)^m$

Decomposability

The accumulation factor $f(t)$ is decomposable if

$$f(x + y) = f(x) \cdot f(y)$$

$$\forall x, y \geq 0$$

Force of Interest

For the financial law $f(t)$ we define the force of interest as

$$\begin{aligned} \rho(t) &\equiv \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{f(t)} \cdot \frac{1}{h} \\ &\equiv \frac{f'(t)}{f(t)} \end{aligned}$$

For simple interest: $\rho(t) = \frac{1}{1+it}$

For compound interest: $\rho(t) = \ln(1+i)$

Cash flow

A cash flow is made by finite (or infinite) sequence of pairs (a_i, t_i) that can be represented on the time-line

If $a_i > 0$ in flow

If $a_i < 0$ outflow

Typical investment: pay once, they you cash in

Typical loan: receive once, they you repay

Annuity: all amounts with the same sign (typically inflows)

DCF - Discounted Cash Flow

Consider the financial operation

years	$t_0 = 0$	t_1	t_2	...	$t_n = T$
CF	a_0	a_1	a_2	...	a_n

The function $g : (-1, +\infty) \rightarrow R$ defined by

$$G(x) = \sum_{s=0}^n \frac{a_s}{(1+x)^{t_s}}$$

Is the discounted cash flow associated to the financial operation.

It describes the present value of the cash flow where the compound annual interest rate is taken as a variable.

NPV - Net Present Value

Consider the financial operation

<i>years</i>	$t_0 = 0$	t_1	t_2	...	$t_n = T$
<i>CF</i>	a_0	a_1	a_2	...	a_n

And its discounted cash flow $G(x)$.

The amount $G(i)$ is called net present value of the financial operation at the rate i .

$$G(i) = \sum_{s=0}^n \frac{a_s}{(1+i)^{t_s}}$$

Internal Rate

The internal rate of a financial operation is any solution of the equation

$$G(x) = 0$$

(i.e. any zero of the discounted cash flow, provided that they exist),

If it's a typical investment: IRR (internal rate of return)

If it's a typical loan: EC (effective cost)

YTM - Yield to Maturity of a zero coupon bond

The yield to maturity (or gross compound yield) of a zero coupon bond is

$$i(0,T) = \left(\frac{N}{P_0}\right)^{1/T} - 1$$

YTM - Yield to Maturity of a bond with coupons

Consider the bond with coupons described by the flow

<i>years</i>	$t_0 = 0$	t_1	t_2	...	$t_n = T$
<i>CF</i>	$-P_0$	a_1	a_2	...	$a_n + R$

The yield to maturity is the internal rate of return of the investment, that is the (unique) solution of the equation

$$G(x) = 0$$

i.e. the value i such that

$$G(i) = -P_0 + \frac{a_1}{(1+i)^{t_1}} + \frac{a_2}{(1+i)^{t_2}} + \dots + \frac{a_n + R}{(1+i)^T} = 0$$

Duration of a cash flow

Consider the investment

<i>years</i>	$t_0 = 0$	t_1	t_2	...	$t_n = T$
<i>CF</i>	$-P_0(0,i)$	a_1	a_2	...	a_n

With price

$$P(0,i) = \sum_{s=1}^n \frac{a_s}{(1+i)^{t_s}}$$

At the market rate i .

We call duration of the investment the average of the future maturities weighted with the percentage of the discounted amounts. In a formula:

$$D = \sum_{s=1}^n t_s \cdot \frac{a_s}{P} \cdot \frac{1}{(1+i)^{t_s}}$$

For a zero coupon bond:

$$D = \text{Maturity}$$

Financial immunization

An investment at rate i is financially immunized at year z^* against the interest rate risk if

$$V(z^*, i + \Delta i) \geq V(z^*, i) \quad \forall \Delta i$$

And the lifetime z^* is called date of immunization

Volatility of the price of the bond

The volatility of the price of a bond is the sensitivity of the price to a change in yield rate::

$$\frac{\Delta P(i)}{P(i)} = \frac{P(i + \Delta i) - P(i)}{P(i)}$$

Expressed in percentage (i.e. relative variation of the price)

Modified Duration

We call modified duration of a security at the market rate i the value

$$D^*(i) = \frac{D(i)}{1 + i}$$

Contingent claim

We call contingent claim any state contingent payoff $\underline{w} \in R^k$

Replicable claim

A claim $\underline{w} \in R^k$ is replicable in the market if there exists a portfolio such that

$$\underline{w} = \sum_{j=1}^n x_j \underline{y}_j$$

Market W

We call market W the vector subspace of R^k made of all replicable claims, i.e.

$$W = \text{span}L$$

Complete Market

The market W is complete if

$$W \equiv R^k$$

That is, if all contingent claims are replicable

Pure contingent claims

We call pure contingent claims the claims in R^k

$$\{\underline{e}_1, \dots, \underline{e}_n\}$$

Where \underline{e}_i pays 1€ if the state ω_i is reached tomorrow and 0 if another state is reached.

Payoff operator

The payoff operator is the function associating to each portfolio \underline{x} the claim \underline{w} that it induces:

$$R : R^n \rightarrow R^k$$

$$\underline{x} \rightarrow R(\underline{x}) = \underline{w} = \sum_{j=1}^n \underline{x}_j \underline{e}_j$$

Market value of the portfolio

The market value of the portfolio \underline{x} is the function associating to each portfolio its price today:

$$v : R^n \rightarrow R$$

$$\underline{x} \rightarrow v(\underline{x}) = \underline{p} \cdot \underline{x} = \sum_{j=1}^n x_j p_j$$

LOP - Law of One Price

The financial market $\{L, \underline{p}\}$ satisfies the LOP if portfolios inducing the same contingent claim share the same market value:

$$\forall \underline{x}^*, \underline{x}^* \quad \text{such that} \quad R(\underline{x}^*) = R(\underline{x}^*) \implies v(\underline{x}^*) = v(\underline{x}^*)$$

Price of the claim

Provided that the LOP is satisfied, we call price $p_{\underline{w}}$ of the claim $\underline{w} \in R^k$ the value of (any) replicating portfolio inducing such claim:

$$p_{\underline{w}} = v(\underline{x}) \quad \text{where } R(\underline{x}) = \underline{w}$$

Pricing kernel

Suppose the financial market $\{L, \underline{p}\}$ satisfies the LOP, then there exists a unique vector $\pi \in W$ such that:

$$f(\underline{w}) = \pi \cdot \underline{w} \quad \forall \underline{w} \in W \subseteq R^k$$

Such vector π is called the pricing kernel of the claims.

Arbitrage

Arbitrage of I kind:
$$\begin{cases} Y \cdot \underline{x} = R(\underline{x}) \geq 0 \\ \underline{p} \bullet \underline{x} = v(\underline{x}) < 0 \end{cases}$$

Arbitrage of II kind:
$$\begin{cases} Y \cdot \underline{x} = R(\underline{x}) > 0 \\ \underline{p} \bullet \underline{x} = v(\underline{x}) \leq 0 \end{cases}$$

FOR DOUBTS OR SUGGESTIONS ON THE HANDOUTS



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