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B.LAB

HANDOUTS

MATHEMATICS MODULE II (APPLIED)

-THEOREMS WITH PROOFS-

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Mathematics Module II (Applied)

Theorems with Proofs

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Probability

Monotonicity of a Measure

Each measure is monotone

$$A \subseteq B \Longrightarrow \mu(B) \geq \mu(A)$$

PROOF

Consider $\quad A \subseteq B$

Then $\quad B = A \cup (B \backslash A)$

A and $B \backslash A$ are incompatible events

Since μ is Additive we get

$$\mu(B) = \mu(A) + \underbrace{\mu(B \backslash A)}_{\geq 0 \text{ since } \mu \text{ is Positive}}$$

$$\mu(B) \geq \mu(A)$$

Relation between the Measure of A∪B and A∩B

For any measure $\mu : 2^{\Omega} \to [0, +\infty)$ there holds

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

For any pair of events A and B of Ω.

PROOF

If $A \cap B = \varnothing$ it's the Additivity.

If not:

$$A \cup B = (A \backslash B) \cup (A \cap B) \cup (B \backslash A)$$

$$\mu(A \cup B) = \mu(A \backslash B) + \mu(A \cap B) + \mu(B \backslash A)$$

Moreover

$$A = (A \backslash B) \cup (A \cap B) \qquad B = (B \backslash A) \cup (A \cap B)$$

$$\mu(A) = \mu(A \backslash B) + \mu(A \cap B) \qquad \mu(B) = \mu(B \backslash A) + \mu(A \cap B)$$

$$\mu(A \backslash B) = \mu(A) - \mu(A \cap B) \qquad \mu(B \backslash A) = \mu(B) - \mu(A \cap B)$$

Leading to

$$\mu(A \cup B) = \mu(A) - \mu(A \cap B) + \mu(A \cap B) + \mu(B) - \mu(A \cap B)$$

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

A Probability Measure is Simple if it exists a finite event E such that P(E)=1

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability measure, where the event E takes all the probability.

$$\forall \text{outcome } \omega \notin E \text{ then } P(\omega) = 0$$

PROOF

$$\omega \notin E \Longrightarrow \omega \in E^c$$

By Positivity and Monotonicity

$$0 \leq P(\omega) \leq P(E^c) = 1 - P(E) = 0$$

Thus $\quad P(\omega) = 0$

Support of a Simple Probability

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability. Then:

1. $SuppP$ is a FINITE event
2. $P(SuppP) = 1$
3. If $P(A) = 1 \implies A \supseteq SuppP$

PROOF

1.

- P is simple $\implies \exists$ finite $E \,|\, P(E) = 1$

  $\omega \in E^c \implies P(\omega) = 0$

- $SuppP = \{\omega \in \Omega \,|\, P(\omega) > 0\}$

Thus $SuppP \subseteq E$ that is $(SuppP)^c \supseteq E^c$ (contrapositive)

Any subset of a finite set is finite.

2.

$$P(SuppP) = \underbrace{P(SuppP \cap E^c)}_{P(\varnothing) = 0} + \underbrace{P(SuppP \cap E)}_{\leq P(E) = 1}$$

By contradiction

$$P(SuppP \cap E) < 1 \quad \text{that would give}$$

$$P((SuppP \cap E)^c) = 1 - P(SuppP \cap E) > 0$$

thus some $\omega \in (SuppP)^c$ would have $P(\omega) > 0$ BUT it's not feasible $\omega \notin SuppP$

3.

By contradiction $A \subsetneq SuppP$

For some $\omega \in SuppP$ and $\omega \notin A : P(\omega) > 0$

Since $P(\omega) > 0$

$$1 = P(A) \underset{\text{contradiction}}{<} P(A) + P(\omega) \underset{\text{additivity}}{=} P(A \cup \{\omega\}) \leq P(\Omega) = 1$$

Finite Reduction of a Simple Probability

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability. Then ∀event A

$$P(A) = \sum_{\omega \in A \cap SuppP} P(\omega)$$

PROOF

Consider $A = (A \cap SuppP) \cup (A \cap (SuppP)^c)$

By Additivity

$$P(A) = P(A \cap SuppP) + \underbrace{P(A \cap (SuppP)^c)}_{=0 \text{ since } \leq P((SuppP)^c) = 0}$$

By Finite Additivity since $SuppP$ is finite

$$P(A) = P(A \cap SuppP) = \sum_{\omega \in A \cap SuppP} P(\omega)$$

Every Simple Probability is a convex L.C. of Dirac Probabilities

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability. Then ∀event A

$$P(A) = \sum_{\omega \in SuppP} P(\omega)\delta_{\omega}(A)$$

PROOF

$$P(A) = \sum_{\omega \in A \cap SuppP} P(\omega) \cdot 1 = \sum_{\omega \in A \cap SuppP} P(\omega)\delta_{\omega}(A) = \sum_{\omega \in SuppP} P(\omega)\delta_{\omega}(A)$$

Countable Additivity for Simple Probabilities

1. Any Dirac probability is countable additive
2. Simple probabilities are countable additive

PROOF

1. Consider the Dirac probability δ_{ω_0} with $\omega_0 \in \Omega$

   Take $A_n \downarrow A$

   - If $\omega_0 \in A \Longrightarrow \omega_0 \in A_n$ (since every A_n includes A)

     $$\delta_{\omega_0}(A) = 1 \quad \text{and} \quad \delta_{\omega_0}(A_n) = 1$$

     Therefore $\quad P(A_n) = 1 \downarrow 1 = P(A)$

   - If $\omega_0 \notin A \Longrightarrow$ surely $\exists M \,|\, \forall n > M \qquad \omega_0 \notin A$

     In particular $\quad \delta_{\omega_0}(A) = 0 \quad \text{and} \quad \delta_{\omega_0}(A_n) = 0 \quad \forall n > M$

     Therefore $\quad P(A_n) = 0 \downarrow 0 = P(A)$

2. Let $P : 2^{\Omega} \to [0,1]$ be a simple probability.

   It is a convex L.C. of Dirac Probabilities.

   $$P(A) = \sum_{\omega \in A \cap SuppP} P(\omega) = \sum_{\omega \in SuppP} P(\omega)\delta_{\omega}(A)$$

   Consider a countable collection of sets $A_n \downarrow A$.

   Since Dirac probability is countable additive

   $$\lim_{n\to+\infty} P(A_n) = \lim_{n\to+\infty} \sum_{\omega \in SuppP} P(\omega)\delta_{\omega}(A) = \sum_{\omega \in SuppP} P(\omega)(\lim_{n\to+\infty} \delta_{\omega}(A_n) = \sum_{\omega \in SuppP} P(\omega)\delta_{\omega}(A) = P(A)$$

   P is countably additive.

A Uniform Probability is NOT Countably Additive.

On the sample space $\Omega = N$ there is

NO COUNTABLY ADDITIVE UNIFORM PROBABILITY $P : 2^N \to [0,1]$

PROOF

By contradiction $\exists$ a COUNTABLY ADDITIVE PROBABILITY $P : 2^N \to R$

set $K = P(n)$ $\forall n \in N$ $K \geq 0$ by POSITIVITY

As $N = \cup_{n \in N} \{n\}$

By countable additivity we reach the contradiction

$$1 = P(N) = P(\cup \{n\}) \overset{\downarrow}{\underset{\uparrow}{=}} \sum_{n \in N} P(n) = K + \ldots + K \ldots + = \begin{cases} 0 \ if \ K = 0 \\ +\infty \ if \ K > 0 \end{cases}$$

Linearity of the Expected Value

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability and $f, g : \Omega \to R$ any pair of random variables and any $\alpha, \beta \in R$. It holds:

$$E_p(\alpha f + \beta g) = \alpha E_p(f) + \beta E_p(g)$$

PROOF

$\forall \alpha, \beta \in R$

$$E_p(\alpha f + \beta g) = \sum_{\omega \in SuppP} (\alpha f(\omega) + \beta g(\omega))P(\omega)$$

$$= \alpha \sum_{\omega \in SuppP} f(\omega)P(\omega) + \beta \sum_{\omega \in SuppP} g(\omega)P(\omega)$$

$$= \alpha E_p(f) + \beta E_p(g)$$

Monotonicity of the Expected Value

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability and $f, g : \Omega \to R$ any pair of random variables.

$$\text{If } f \geq g \qquad \Longrightarrow \quad E_p(f) \geq E_p(g)$$

PROOF

Consider $\quad f(\omega) \geq g(\omega)\ \forall \omega \in \Omega$

Multiplying both sides by $P(\omega) \geq 0$

$$f(\omega)P(\omega) \geq g(\omega)P(\omega) \qquad \forall \omega \in \Omega \text{ and } \forall \omega \in SuppP$$

Adding over the $SuppP$ we get

$$E_p(f) = \sum_{\omega \in SuppP} f(\omega)P(\omega) \geq \sum_{\omega \in SuppP} g(\omega)P(\omega) = E_p(g)$$

Extension to any finite set that contains the Support (Expected Value)

Let $P : 2^\Omega \to [0,1]$ be a simple probability and $f : \Omega \to R$ a random variable.

$$\text{If } A \supseteq SuppP \quad \Longrightarrow \quad E_p(f) = \sum_{\omega \in A} f(\omega)P(\omega)$$

PROOF

$$E_p(f) = \sum_{\omega \in SuppP} f(\omega)P(\omega) = \sum_{\omega \in A} f(\omega)P(\omega) \quad \text{since } P(\omega) = 0 \ \forall \omega \in A \backslash SuppP$$

Characterization for Simple Probabilities of Random Variables that are P-equal a.e.

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability and $f, g : \Omega \to R$ 2 random variables.

Then:

f, g are P-equal a.e. $\Longleftrightarrow \forall \omega \in SuppP$ we have $f(\omega) = g(\omega)$

PROOF

$\Longleftarrow$

We have that $f(\omega) = g(\omega) \qquad \forall \omega \in SuppP$

Therefore

$$SuppP \subseteq \{\omega \in \Omega : f(\omega) = g(\omega)\}$$

From Monotonicity

$$1 = P(SuppP) \leq P(f = g) \Longrightarrow P(f = g) = 1$$

And f and g coincide a.e. according to P.

$\Longrightarrow$

For a simple probability

$$SuppP \subseteq A \qquad \forall A \subseteq \Omega \quad P(A) = 1$$

As a set A with $P(A) = 1$

we take the set where $f = g$

Therefore, we deduce that

f and g coincide on the $SuppP$

P-equal a.e. Random Variables have the same Expected Value

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability and $f, g : \Omega \to R$ 2 random variables P-equal a.e.

Then:

$$E_p(f) = E_p(g)$$

PROOF

$$E_p(f) = \sum_{\omega \in SuppP} f(\omega)P(\omega) = \sum_{\omega \in SuppP} g(\omega)P(\omega) = E_p(g)$$

Since f and g coincide on the $SuppP$

Computation Formula for the Variance

Let $P : 2^{\Omega} \rightarrow [0,1]$ be a simple probability. For each random variable $f : \Omega \rightarrow R$

$$V_p(f) = E_p(f^2) - [E_p(f)]^2$$

PROOF

Definition $V_p(f) = E_p(f - E_p(f))^2$

I multiply

$$= E_p(f^2 - 2fE_p(f) + [E_p(f)]^2$$

Linearity of Expected Value

$$= E_p(f^2) - 2E_p(f)E_p(f) + [E_p(f)]^2$$

Simplifying

$$= E_p(f^2) - [E_p(f)]^2 + [E_p(f)]^2$$

$$V_p(f) = E_p(f^2) - [E_p(f)]^2$$

Computation Formula for the Covariance

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability. For each random variables $f, g : \Omega \to R$

$$COV_P(f, g) = E_p(f \cdot g) - E_p(f) \cdot E_p(g)$$

PROOF

Definition $\quad COV_P(f, g) = E_p(f - E_p(f)) \cdot (g - E_p(g))$

I multiply

$$= E_p(fg - fE_p(g) - gE_p(f) + E_p(f)E_p(g))$$

Linearity of Expected Value

$$= E_p(f \cdot g) - E_p(f)E_p(g) - E_p(g)E_p(f) + E_p(f)E_p(g)$$

Simplifying:

$$COV_P(f, g) = E_p(f \cdot g) - E_p(f) \cdot E_p(g)$$

Expectation of Affine Functions of Random Variables

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability. For each random variable $f : \Omega \to R$ and each $\alpha, \beta \in R$

$$E_p(\alpha f + \beta) = \alpha E_p(f) + \beta$$

PROOF

Definition $$E_p(\alpha f + \beta) = \sum_{\omega \in SuppP} (\alpha f(\omega) + \beta) P(\omega)$$

$$= \sum_{\omega \in SuppP} (\alpha f(\omega) P(\omega) + \beta P(\omega))$$

$$= \sum_{\omega \in SuppP} \alpha f(\omega) P(\omega) + \sum_{\omega \in SuppP} \beta P(\omega)$$

$$= \alpha \sum_{\omega \in SuppP} f(\omega) P(\omega) + \beta \sum_{\omega \in SuppP} P(\omega)$$

$$E_p(\alpha f + \beta) = \alpha E_p(f) + \beta$$

Variance of Affine Functions of Random Variables

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability. For each random variable $f : \Omega \to R$ and each $\alpha, \beta \in R$

$$V_p(\alpha f + \beta) = \alpha^2 V_p(f)$$

PROOF

Definition: $V_p(\alpha f + \beta) = E_p(\alpha f + \beta - E_p(\alpha f + \beta))^2$

Linearity of Expected Value

$$= E_p(\alpha f + \beta - \alpha E_p(f) - \beta)^2$$

$$= E_p(\alpha^2(f - E_p(f))^2)$$

$$= \alpha^2 E_p(f - E_p(f))^2$$

$$V_p(\alpha f + \beta) = \alpha^2 V_p(f)$$

Covariance of Affine Functions of Random Variables

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability. For each random variables $f, g : \Omega \to R$ and each $\alpha, \beta, \gamma, \delta \in R$. Then:

$$COV_P(\alpha f + \beta, \gamma g + \delta) = \alpha\gamma COV_P(f, g)$$

PROOF

Definition $\quad COV_P(\alpha + \beta, \gamma g + \delta) = E_p(\alpha f + \beta - E_p(\alpha f + \beta)) \cdot (\gamma g + \delta - E_p(\gamma g + \delta))$

Linearity of Expected Value

$$= E_p(\alpha(f - E_p(f)) \cdot \gamma(g - E_p(g)))$$

$$= \alpha\gamma E_p(f - E_p(f))(g - E_p(g))$$

$$COV_P(\alpha f + \beta, \gamma g + \delta) = \alpha\gamma COV_P(f, g)$$

Variance of a Sum of Random Variables

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability. For all random variables $f, g : \Omega \to R$

$$V_p(f+g) = V_p(f) + V_p(g) + 2COV_P(f,g)$$

PROOF

Definition $\quad V_p(f+g) = E_p((f+g) - E_p(f+g))^2$

Linearity of Expected Value

$$= E_p(f - E_p(f) + g - E_p(g))^2$$

$$= E_p[(f - E_p(f))^2 + (g - E_p(g))^2 + 2(f - E_p(f))(g - E_p(g)]$$

$$= E_p[(f - E_p(f))]^2 + E_p[(g - E_p(g))]^2 + 2E_p[(f - E_p(f))(g - E_p(g)]$$

Linearity of Expected Value

$$V_p(f+g) = V_p(f) + V_p(g) + 2COV_P(f,g)$$

Boundedness Property for the Covariance

Let $P : 2^{\Omega} \to [0,1]$ be a simple probability. For all random variables $f, g : \Omega \to R$

$$|COV_P(f,g)| \leq \sigma_p(f)\sigma_p(g)$$

PROOF

Definition

$$|COV_P(f,g)| = |\sum_{\omega \in SuppP} (f(\omega) - E_p(f))(g(\omega) - E_p(g))P(\omega)|$$

Consider $\quad E_p(f) = 0 = E_p(g)$

$$= |\sum_{\omega \in SuppP} f(\omega)g(\omega)P(\omega)|$$

$$= |\sum_{i=1}^{n} f(\omega_i)g(\omega_i)P(\omega_i)|$$

For Cauchy-Schwarz inequality in R^n $\quad \forall \underline{u}, \underline{v} \in R^n \quad |\underline{u} \bullet \underline{v}| \leq ||\underline{u}|| \cdot ||\underline{v}||$

$$= |\sum_{i=1}^{n} f(\omega_i)\sqrt{P(\omega_i)}g(\omega_i)\sqrt{P(\omega_i)}| \leq \underbrace{\sqrt{\sum_{i=1}^{n} f^2(\omega_i)P(\omega_i)}}_{=\sqrt{V_p(f)}} \cdot \underbrace{\sqrt{\sum_{i=1}^{n} g^2(\omega_i)P(\omega_i)}}_{=\sqrt{V_p(g)}}$$

Now consider any f, g without 0 expectation

$$\tilde{f} = f - E_p(f) \qquad \text{and} \qquad \tilde{g} = g - E_p(g)$$

By Linearity

$$E_p(\tilde{f}) = 0 \qquad \text{and} \qquad E_p(\tilde{g}) = 0$$

$$|COV_P(\tilde{f}, \tilde{g})| \leq \sqrt{V_p(\tilde{f})}\sqrt{V_p(\tilde{g})}$$

Denote

$$\beta = E_p(f) \qquad \text{and} \qquad \delta = E_p(g)$$

$$COV_P(\tilde{f}, \tilde{g}) = COV_P(f - \beta, g - \delta) = COV_P(f, g)$$

$$V_p(\tilde{f}) = V_p(\tilde{f} - \beta) = V_p(f)$$

$$V_p(\tilde{g}) = V_p(\tilde{g} - \delta) = V_p(g)$$

Leading to

$$\sigma_p(\tilde{f}) = \sigma_p(f)$$

$$\sigma_p(\tilde{g}) = \sigma_p(g)$$

Providing

$$\begin{array}{ccccc} |COV_P(\tilde{f}, \tilde{g})| & \leq & \sqrt{V_p(\tilde{f})} & & \sqrt{V_p(\tilde{g})} \\ \| & & \| & & \| \\ |COV_P(f, g)| & \leq & \sigma_p(f) & \cdot & \sigma_p(g) \end{array}$$

The Distribution Function is Increasing

The cumulative distribution function of the random variable $f : \Omega \to R$ under the probability $P : 2^{\Omega} \to [0,1]$ increasing. That is, the function:

$$\Phi : R \to [0,1]$$

$$x \to \Phi(x) = P(f \leq x)$$

Is $|\forall x_1 < x_2 \Longrightarrow \Phi(x_1) \leq \Phi(x_2)$

PROOF

From the Monotonicity of the probability:

$$A \subseteq B \Longrightarrow P(A) \leq P(B)$$

Let $x_1 \leq x_2$

The lower level sets are included one in the other

$$(f \leq x_1) \subseteq (f \leq x_2)$$

hence $\quad P(f \leq x_1) \leq P(f \leq x_2)$

Deducing $\quad \Phi(x_1) \leq \Phi(x_2)$

The Distribution Function of an Essentially Bounded Random Variable is Eventually Bounded

Let $f : \Omega \to R$ be an essential bounded random variable under the probability $P : 2^{\Omega} \to [0,1]$. Then $\exists$ 2 scalars $a, b \in R \,|$

$$\Phi(x) = 0 \quad \forall x \leq a \quad and \quad \Phi(x) = 1 \quad \forall x \geq b$$

PROOF

Since f is essentially bounded, $\exists m, M \ \ | \ \ P(m \leq f \leq M) = 1.$ Note that:

- If $x < m$ $\quad (f \leq x) \cap (m \leq f \leq M) = \varnothing$ $\quad$ hence $\quad \Phi(x) = 0$
- If $x \geq M$ $\quad (m \leq f \leq M) \subseteq (f \leq x)$ $\quad$ by monotonicity $\quad \Phi(x) = 1$

Take $b = M$ and as a any value $< m$ to get the property.

If a Distribution Function with a Carrier [a,b] has a continuous Density Function, then it is equal to zero for every x that does not belong to the Carrier

Let Φ be a distribution function with a carrier $[a, b]$

If Φ has a continuous density φ, then

$$\varphi(x) = 0 \quad \forall x \notin [a, b]$$

PROOF

By definition $\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt$

Consider any $z > b$.

Since all the probability is from a to b ($\int_{a}^{b} \varphi(t)dt = 1$)

$$\int_{b}^{z} \varphi(t)dt = 0$$

A non-negative continuous function with zero integral on $[b, z]$ is the **zero function** and the arbitrarily of z provides the result $\forall x > b$.

So $$\varphi(x) = 0 \quad \forall x \in [b, z]$$

A similar argument can be used to prove the thesis as $x < a$.

A Distribution Function has a unique continuous Density Function if and only if it is continuously differentiable

Let Φ be a distribution function with a carrier $[a, b]$

$$\Phi \text{ has a unique continuous density } \varphi \Longleftrightarrow \Phi \text{ is continuously differentiable (and } \Phi' = \varphi)$$

PROOF

Immediate reformulation of Torricelli-Barrow Theorem with $\gamma = \Phi'$

(Note that $\Phi(a) = \int_{-\infty}^{a} \varphi(t)dt = \int_{a}^{a} \varphi(t)dt = 0$)

Expected Value of a Random Variable with respect to Simple Probabilities as a Stieltjes Integral

Let $P : 2^{\Omega} \to R$ be a simple probability. For each random variable $f : \Omega \to R$

$$E_p(f) = \int_a^b x d\Phi(x)$$

(where $[a, b]$ is any carrier of the random variable)

PROOF

Since P is a simple probability

$$SuppP = \{\omega_1, \omega_2, \ldots, \omega_n\}$$

Denote $x_i = f(\omega_i)$

Without loss of generality we can assume that these values are distinct and

$$x_1 < x_2 < x < \ldots < x_n;$$

Moreover they all belong to the carrier, that is

$x_1 > a$ and $x_n < b$.

$$P(\omega_i) = P(f = x_i)$$

$$E_p(f) = \sum_{\omega \in SuppP} f(\omega)P(\omega) = \sum_{i=1}^{n} x_i P(f = x_i)$$

Recall that

$$P(f = x_i) = \Phi(x_i) - \Phi(x_i^-)$$

Using the last theorem about Stieltjes integral we conclude:

$$E_p(f) = \sum x_i[\Phi(x_i) - \Phi(x_i^-)] = \int_a^b x d\Phi(x)$$

Financial Calculus

Necessary and Sufficient Condition for the Decomposability

Let $f(t)$ be an accumulation factor, with $t \geq 0$. Then

f is decomposable $\Longleftrightarrow$ f is a COMPOUND ACCUMULATION LAW

PROOF

$\Longrightarrow$

f is decomposable

$$f(t_1 + t_2) = f(t_1) \cdot f(t_2)$$

The characterization of the functions which satisfy the multiplicative equation provides that:

$f(t) = e^{mt} = e^{\delta t}$, that is the COMPOUND LAW written as a natural log.

$\Longleftarrow$

Consider the COMPOUND LAW written as a natural log

$$f(t) = e^{\delta t}$$

Consider $t_1, t_2 \geq 0$:

$$f(t_1 + t_2) = e^{\delta(t_1+t_2)} = e^{\delta t_1 + \delta t_2} = e^{\delta t_1} \cdot e^{\delta t_2} = f(t_1) \cdot f(t_2)$$

f is decomposable

Formula on Ordinary Annuity with constant installment (PV)

The present value of an ordinary annuity with annual compound rate i is:

$$V_0 = R \cdot a_{n\urcorner i}$$

Where R is the installment and n their number.

PROOF

$$V_0 = R(\sum_{k=1}^{n} \frac{1}{(1+i)^k})$$

$$= \frac{R}{1+i} + \frac{R}{(1+i)^2} + \ldots + \frac{R}{(1+i)^n}$$

Call $z = \dfrac{1}{1+i}$

$$= Rz^1 + Rz^2 + \ldots + Rz^n$$

$$= Rz(z^0 + \ldots + z^{n-1})$$

Recall the Geometric Sum: $\displaystyle\sum_{k=0}^{n-1} z^k = (z^0 + \ldots + z^{n-1}) = \frac{1-z^n}{1-z}$

$$\implies \quad = Rz \cdot \frac{1-z^n}{1-z}$$

Substitute $z = \dfrac{1}{1+i}$

$$= R\frac{1-(\frac{1}{1+i})^n}{i} = R\frac{1-(1+i)^{-n}}{i}$$

Call the fraction $a_{n\urcorner i}$

$$V_0 = R \cdot a_{n\urcorner i}$$

Formula on Ordinary Annuity with constant installment (FV)

The final value of an ordinary annuity with annual compound rate i is:

$$V_T = R \cdot s_{n\neg i}$$

Where R is the installment and n their number.

PROOF

$$V_T = \sum_{k=0}^{n-1} R(1+i)^k$$

$$= R(1+i)^{n-1} + R(1+i)^{n-2} + \ldots + R(1+i) + R$$

Call $z = (1+i)$

$$= Rz^{n-1} + \ldots + Rz^0$$

$$= Rz^0 + \ldots + Rz^{n-1}$$

Recall the Geometric Sum: $\displaystyle\sum_{k=0}^{n-1} z^k = (z^0 + \ldots + z^{n-1}) = \frac{1-z^n}{1-z}$

$$\implies \quad = R \cdot \frac{1-z^n}{1-z}$$

Substitute $z = \dfrac{1}{1+i}$

$$= R\frac{1-(1+i)^n}{1-(1+i)} = R\frac{1-(1+i)^n}{i}$$

Call the fraction $s_{n\neg i}$

$$V_T = R \cdot s_{n\neg i}$$

Formula on Due Annuity with constant installment (PV)

The present value of a due annuity with annual compound rate i is:

$$V_0 = R \cdot \ddot{a}_{n\neg i}$$

Where R is the installment and n their number.

PROOF

$$\ddot{V}_0 = R(\sum_{k=0}^{n-1} \frac{1}{(1+i)^k})$$

$$= R + \frac{R}{1+i} + \frac{R}{(1+i)^2} + \ldots + \frac{R}{(1+i)^{n-1}}$$

Call $z = \dfrac{1}{1+i}$

$$= Rz^0 + Rz^1 + \ldots + Rz^{n-1}$$

Call $a_{n\neg i} = \dfrac{1-(1+i)^{-n}}{i}$

$$= R \cdot \frac{a_{n\neg i}}{z}$$

$$= R \cdot a_{n\neg i} \cdot (1+i)$$

Call $\ddot{a}_{n\neg i} = (1+i) \cdot a_{n\neg i}$

$$V_0 = R \cdot \ddot{a}_{n\neg i}$$

Formula on Due Annuity with constant installment (FV)

The final value of a due annuity with annual compound rate i is:

$$V_T = R \cdot \ddot{s}_{n\neg i}$$

Where R is the installment and n their number.

PROOF

$$\ddot{V}_T = R[\sum_{k=1}^{n}(1+i)^k]$$

$$= R(1+i)^n + \ldots + R(1+i)$$

Call $z = (1+i)$

$$= Rz^n + \ldots + Rz^1$$

$$= Rz^1 + \ldots + Rz^n$$

Call $s_{n\neg i} = \dfrac{1-(1+i)^n}{i}$

$$= R \cdot s_{n\neg i} \cdot z$$

$$= R \cdot s_{n\neg i} \cdot (1+i)$$

Call $\ddot{s}_{n\neg i} = (1+i) \cdot s_{n\neg i}$

$$\ddot{V}_T = R \cdot \ddot{s}_{n\neg i}$$

Duration as a Date of Financial Immunization

Give a fixed income bond that pays

$years$	t_1	t_2	$\dots$	t_n
CF	a_1	a_2	$\dots$	a_n

Assuming that

- The yield to maturity (YTM) of the bond at $t_0 = 0$ is i
- The rate i undergoes a variation Δi at $\tau \in (0, t_1]$ and does not undergo further variations (interest rate risk)

Then, the investment is financially immunized against the interest rate risk at the date

$$z^* = D(i)$$

Where $D(i)$ is the duration of the bond evaluated at the rate i.

PROOF

It's a minimum problem: $\min\limits_{with\ respect\ to\ x} V(z, x)$

Fermat: we want i to be (a minimizer) stationary point.

$$\frac{\partial V}{\partial x}|_{x=i} = 0$$

$$V(z, x) = \sum_{k=1}^{n} a_k (1+x)^{z-t_k}$$

$$\frac{\partial V}{\partial x} = \sum_{k=1}^{n} a_k (z - t_k)(1+x)^{z-t_k-1}$$

$$\frac{\partial V}{\partial x}|_{x=i} = 0 \quad \text{hence} \quad \sum_{k=1}^{n} a_k (z - t_k)(1+i)^{z-t_k-1} = 0$$

Deducing $\displaystyle\sum_{k=1}^{n} a_k z (1+i)^{z-t_k-1} = \sum_{k=1}^{n} a_k \cdot t_k (1+i)^{z-t_k-1}$

$$\frac{\sum_{k=1}^{n} a_k z (1+i)^{-t_k}}{(1+i)^{1-z}} = \frac{\sum_{k=1}^{n} a_k \cdot t_k (1+i)^{-t_k}}{(1+i)^{1-z}} \quad \Longrightarrow \quad z = \frac{\sum_{k=1}^{n} a_k t_k (1+i)^{-t_k}}{\sum_{k=1}^{n} a_k (1+i)^{-t_k}} = D$$

i is also minimum because

$$\frac{\partial^2 V}{\partial^2 x}|_{x=i} > 0$$

Bond Price Volatility

Given fixed income bond that pays

$years$	t_1	t_2	$\dots$	t_n
CF	a_1	a_2	$\dots$	a_n

With the price at the market rate i that is:

$$P(i) = V(0,i) = \sum_{s=1}^{n} \frac{a_s}{(1+i)^{t_s}}$$

Then:

1. $\dfrac{P'(i)}{P(i)} = -\dfrac{D(i)}{1+i}$ (differential form of the volatility)
2. $\dfrac{\Delta P(i)}{P(i)} \approx \dfrac{P'(i)}{P(i)}\Delta i = -\dfrac{D(i)}{1+i}\Delta i$

PROOF

1. $$P(i) = V(0,i) = \sum_{k=1}^{n} a_k \cdot \frac{1}{(1+i)}t_k$$

$$P'(i) = \sum_{k=1}^{n} a_k(-t_k)(1+i)^{-t_k-1}$$

$$= \frac{-1}{1+i}\sum_{k=1}^{n} a_k t_k (1+i)^{-t_k}$$

divide by $P(i)$ $\quad \dfrac{P'(i)}{P(i)} = -\dfrac{D(i)}{1+i}$

2. Modified duration: $\dfrac{D(i)}{1+i} = D^*(i)$

The volatility of the price of a bond equal the modified duration changed in sign.

$$\frac{P'(i)}{P(i)} = -D^*$$

Using the linear approximation of the derivative $P'(i) \approx \dfrac{\Delta P}{\Delta i}$ from $\dfrac{P'(i)}{P(i)} = -D^*$ we deduce:

$$\frac{\Delta P}{\Delta i}\frac{1}{P} \approx -D^*$$

$$\frac{\Delta P}{P} \approx -D^* \cdot \Delta i$$

Linearity of the Pricing Rule

Suppose the financial market $(L, \underline{p})$ satisfies the LOP, then the pricing rule

$$f : W \rightarrow R$$

$$\underline{w} \rightarrow f(\underline{w}) = p_{\underline{w}} = v(\underline{x})$$

is a linear functional.

Which means:

$$f(\alpha \underline{w_1} + \beta \underline{w_2}) = \alpha f(\underline{w_1}) + \beta f(\underline{w_2})$$

PROOF

since $R(\underline{x}) = \underline{w}$ $\quad f(\alpha \underline{w_1} + \beta \underline{w_2}) = f(\alpha R(\underline{x_1}) + \beta R(\underline{x_2}))$

Since R is linear: $\quad = f(R(\alpha \underline{x_1} + \beta \underline{x_2}))$

Since $f(R(\underline{x})) = v(\underline{x})$: $\quad = v(\alpha \underline{x_1} + \beta \underline{x_2})$

Since v is linear: $\quad = \alpha v(\underline{x_1}) + \beta v(\underline{x_2})$

Since $v(\underline{x}) = p_{\underline{w}}$ $\quad = \alpha P_{\underline{w_1}} + \beta P_{\underline{w_2}}$

Since $p_{\underline{w}} = f(\underline{w})$ $\quad = \alpha f(\underline{w_1}) + \beta f(\underline{w_2})$

So we proved that: $\quad f(\alpha \underline{w_1} + \beta \underline{w_2}) = \alpha f(\underline{w_1}) + \beta f(\underline{w_2})$

No Arbitrage Condition of I Kind implies the LOP

A financial market that has no arbitrages of I kind satisfies the LOP

PROOF

Without arbitrage of I kind we have:

- $$R(\underline{x}) \geq 0 \Longrightarrow v(\underline{x}) \geq 0 \forall \underline{x} \in R^n$$

R is linear, v is linear; using $-\underline{x}$ we get

$$-R(\underline{x}) \geq \underline{0} \Longrightarrow -v(\underline{x}) \geq 0 \forall \underline{x} \in R^n$$

- $$R(\underline{x}) \leq \underline{0} \Longrightarrow v(\underline{x}) \leq 0$$

Thus $\qquad R(\underline{x}) = \underline{0} \Longrightarrow v(\underline{x}) = 0$

(Every element in the kernel has zero value)

The linearity of R and v allows to conclude; in fact consider 2 portfolios inducing the same claim

$\forall \underline{x}^*, \underline{x}^\bullet, \in R^n \qquad R(\underline{x}^*) = R(\underline{x}^\bullet) \qquad$ we get by linearity of R

$$R(\underline{x}^* - \underline{x}^\bullet) = \underline{0}$$

No arbitrage I $\qquad v(\underline{x}^* - \underline{x}^\bullet) = 0$

hence $\qquad v(\underline{x}^*) = v(\underline{x}^\bullet)$

$\underline{x}^*$ and $\underline{x}^\bullet$ share the same market value.

Linear and Increasing Pricing Rule

Consider a complete financial market $(L, \underline{p})$ with $\underline{p} \neq \underline{0}$

It satisfies no arbitrage condition of I kind $\iff$ the pricing rule is linear and increasing i.e. $\exists$ a unique pricing kernel $\pi \geq 0 \,|\, f(\underline{w}) = \pi \bullet \underline{w}$

PROOF

$\Longleftarrow$

Consider $\quad R(\underline{x}) \geq 0 \quad$ and $\quad \underline{\pi} \geq 0$

Then $\quad v(\underline{x}) = f(R(\underline{x})) = \underline{\pi} \bullet R(\underline{x}) \geq 0$

Which is: no arbitrage condition of I kind.

$\Longrightarrow$

From the completeness of the market $\exists\, \underline{x}^\bullet \in R^n$, a portfolio inducing the claim $\quad \underline{w} = R(\underline{x}^\bullet)$

Since arbitrage I kind $\quad R(\underline{x}^\bullet) = \underline{w} \geq \underline{0} \quad \Longrightarrow \quad v(\underline{x}^\bullet) \geq 0$

By positivity of f $\quad f(R(\underline{x}^\bullet)) = f(\underline{w}) \geq 0$

Since $\quad f(\underline{w}) = \pi \bullet \underline{w} \quad$ and we have that $w \geq 0$ and $f(w) \geq 0$

Then $\quad \underline{\pi} \geq \underline{0}$

In particular $\quad \forall i = 1,\ldots,k$

$$\underline{e}_i > 0 \quad \Longrightarrow \quad \underbrace{f(\underline{e}_i)}_{=\,\pi_i} > 0$$

Fundamental Theorem of Finance

Consider a complete financial market $(L, \underline{p})$ with $\underline{p} \neq 0$

It satisfies no arbitrage condition of I,II kind $\iff$ the pricing rule is linear and strictly increasing i.e. $\exists$ a unique pricing kernel $\underline{\pi} \gg \underline{0} \,|\, f(\underline{w}) = \pi \bullet \underline{w}$

PROOF

$\Longleftarrow$

I just need to prove that arbitrage condition of II kind doesn't hold.

Consider $R(\underline{x}) > 0$ and $\underline{\pi} \gg 0$ $(\pi_i > 0 \quad \forall i = 1,\ldots,k)$

Then $v(\underline{x}) = f(R(\underline{x})) = \underline{\pi} \bullet R(\underline{x}) > 0$

Which is: no arbitrage condition of II kind.

$\Longrightarrow$

From the completeness of the market $\exists\, \underline{x}^\bullet \in R^n$, a portfolio inducing the claim $\underline{w} = R(\underline{x}^\bullet)$

Since arbitrage I kind $R(\underline{x}^\bullet) = \underline{w} > \underline{0} \implies v(\underline{x}^\bullet) > 0$

$\|$

By positivity of f $f(R(\underline{x}^\bullet)) = f(\underline{w}) > 0$

Since $f(\underline{w}) = \pi \bullet \underline{w}$ and we have that $w > 0$ and $f(w) > 0$

Then $\underline{\pi} \gg \underline{0}$

In particular $\forall i = 1,\ldots,k$

$$\underline{e}_i > 0 \implies f(\underline{e}_i) > 0$$

$\|$

π_i

FOR DOUBTS OR SUGGESTIONS ON THE HANDOUTS

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