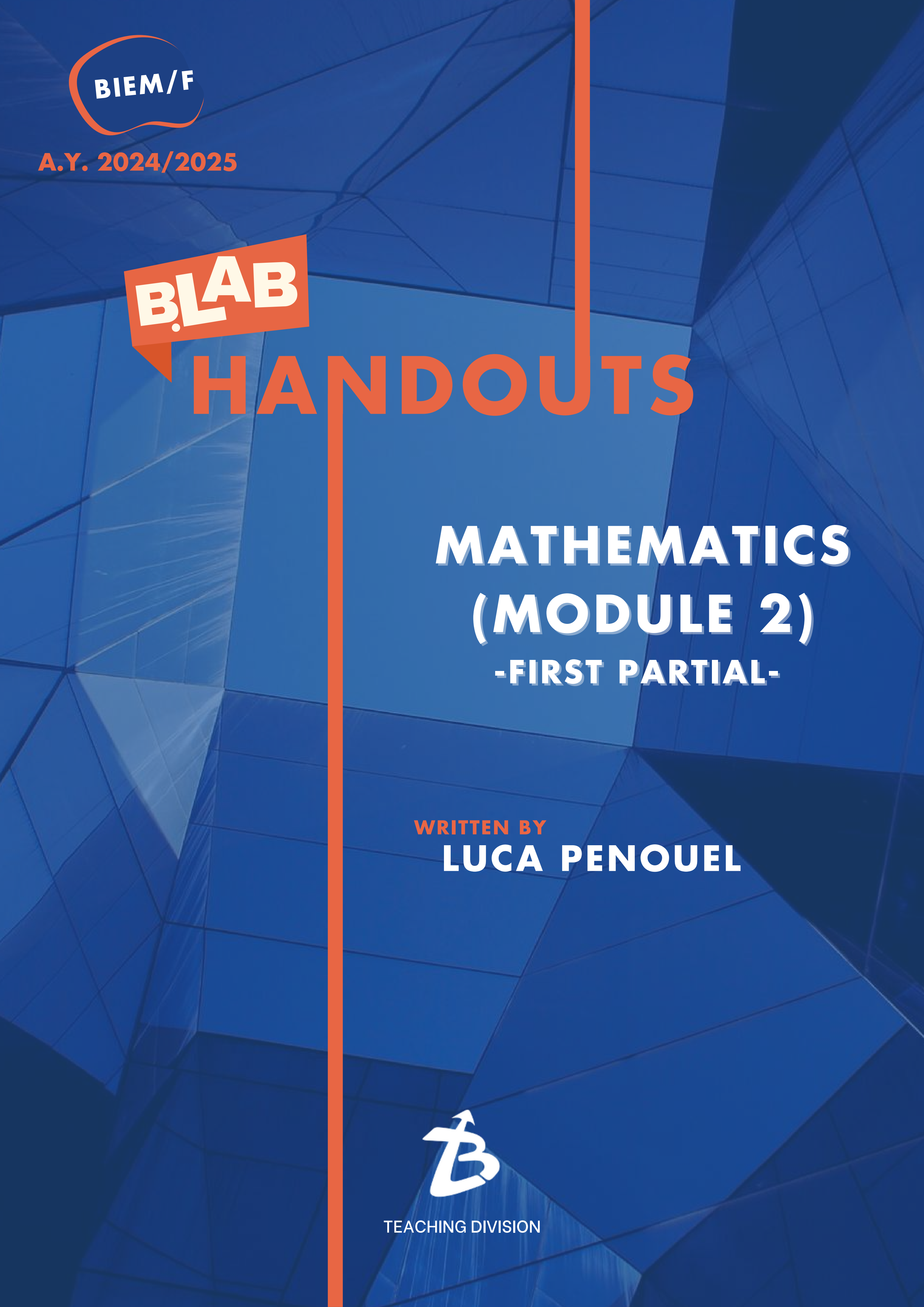

BIEM/F

A.Y. 2024/2025

B.LAB
HANDOUTS

MATHEMATICS (MODULE 2)
-FIRST PARTIAL-

WRITTEN BY
LUCA PENOUEL

TEACHING DIVISION

Linear Algebra

Proofs, Definitions and Theorems

(2024-2025)

These notes have been compiled by Luca Penouel as an additional resource to aid BIEM and BIEF students of the **Mathematics 2 course (30063)**.

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Updated: February 18, 2025

* = IN CLASS
H = HOMEWORK

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PROOFS

1) CHARACTERISTIC POLYNOMIAL AND EIGENVALUES *

STATEMENT: CONSIDER A SYMMETRIC MATRIX A OF ORDER n, A SCALAR $\lambda \in \mathbb{R}$ IS AN EIGENVALUE OF $A \iff \det(A - \lambda I_n) = 0$, WHERE I_n IS THE IDENTITY MATRIX

PROOF: BY DEFINITION $\lambda \in \mathbb{R}$ IS AN EIGENVALUE OF $A \iff A\underline{x} = \lambda \underline{x}$ WITH $\underline{x} \neq \underline{0}$. WE REWRITE AS $(A - \lambda I_n)\underline{x} = \underline{0}$. LET $(A - \lambda I_n) = B$. $B\underline{x} = \underline{0}$ IS AN HOMOGENEUS LINEAR SYSTEM AND ADMITS NON ZERO SOLUTIONS $\iff \det(B) = 0$ SO, $\iff \det(A - \lambda I) = 0$

2) EIGENSPACES *

STATEMENT: CONSIDER A SYMMETRIC MATRIX A OF ORDER n, IF $\lambda \in \sigma(A)$, THEN THE SET $W_\lambda = \{\underline{x} \in \mathbb{R}^n : (A - \lambda I_n) \cdot \underline{x} = \underline{0}\}$ IS A VECTOR SUBSPACE OF $\mathbb{R}^n$, CALLED EIGENSPACE.

PROOF: LET $\lambda \in \sigma(A)$, CONSIDER $\underline{x_1}, \underline{x_2} \in W_\lambda \Rightarrow (A - \lambda I)\underline{x_1} = \underline{0}$ AND $(A - \lambda I)\underline{x_2} = \underline{0}$. NOW TAKE $\alpha \underline{x_1} + \beta \underline{x_2}$, WHERE $\alpha, \beta \in \mathbb{R}$. WE HAVE $(A - \lambda I)(\alpha \underline{x_1} + \beta \underline{x_2}) = \alpha (A - \lambda I)\underline{x_1} + \beta (A - \lambda I)\underline{x_2} = \alpha \cdot \underline{0} + \beta \cdot \underline{0} = \underline{0}$. $\Rightarrow \alpha \underline{x_1} + \beta \underline{x_2} \in W_\lambda$, SO W_λ IS A VECTOR SUBSPACE

3) ORTHOGONAL EIGENSPACES*

STATEMENT:
FOR ALL EIGENPAIRS OF $(\lambda\ \underline{x})$ AND $(\lambda', \underline{x}')$ OF A,
WE HAVE: IF $\lambda \neq \lambda' \implies \underline{x} \perp \underline{x}'$

PROOF: TAKE EIGENPAIRS $(\lambda, \underline{x})$ AND $(\lambda', \underline{x}')$, WHERE $\lambda \neq \lambda'$.
WE KNOW $A\underline{x} = \lambda \underline{x}$ AND $A\underline{x}' = \lambda' \underline{x}'$. WE HAVE:
(1) $A\underline{x} \cdot \underline{x}' = \lambda \underline{x} \cdot \underline{x}' = \lambda(\underline{x} \cdot \underline{x}')$ AND WE ALSO HAVE
(2) $A\underline{x} \cdot \underline{x}' = \underline{x} \cdot A^T \cdot \underline{x}' = \underline{x} \cdot A \cdot \underline{x}' = \underline{x} \cdot \lambda' \cdot \underline{x}' = \lambda' \cdot (\underline{x} \cdot \underline{x}')$
(1) + (2) $\implies \lambda(\underline{x} \cdot \underline{x}') = \lambda'(\underline{x} \cdot \underline{x}') \implies (\lambda - \lambda') \cdot (\underline{x} \cdot \underline{x}') = 0$.
AND SINCE $\lambda \neq \lambda'$ WE HAVE $\underline{x} \cdot \underline{x}' = 0 \implies \underline{x} \perp \underline{x}'$

4) EIGENVALUES OF THE INVERSE MATRIX*

STATEMENT: i) A SYMMETRIC MATRIX IS INVERTIBLE $\iff$ ALL OF ITS EIGENVALUES ARE NON-ZERO, AND IN THAT CASE:
ii) $\lambda \in \sigma(A) \iff \frac{1}{\lambda} \in \sigma(A^{-1})$

PROOF: i) $0 \in \sigma(A) \iff \det(A - 0 \cdot I) = 0 \iff \det(A) = 0$
SO $\det(A) \neq 0$ i.e. IT IS INVERTIBLE $\iff 0 \notin \sigma(A)$.

ii) LET A BE INVERTIBLE, AND $\lambda \in \sigma(A)$, $\lambda \neq 0$ AND CONSIDER $\underline{x} \in \mathbb{R}^n$ WHICH IS THE EIGENVECTOR ASSOCIATED WITH λ. $A\underline{x} = \lambda \underline{x}$ IMPLIES $\underline{x} = \frac{1}{\lambda} A \underline{x}$. NOW TAKE $A^{-1} \cdot \underline{x} \implies A^{-1}\underline{x} = A^{-1} \cdot \frac{1}{\lambda} \cdot A\underline{x} = \frac{1}{\lambda} A^{-1} \cdot A \cdot \underline{x} = \frac{1}{\lambda}\underline{x}$. SO WE NOW HAVE $A^{-1}\underline{x} = \frac{1}{\lambda}\underline{x} \implies \frac{1}{\lambda} \in \sigma(A^{-1})$.
WE HAVE THAT THE CONVERSE IS ALSO TRUE, SINCE $A = (A^{-1})^{-1}$.

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5) POSITIVE DEFINITE MATRIX ARE INVERTIBLE*

STATEMENT: i) A POSITIVE DEFINITE MATRIX IS INVERTIBLE AND ii) ITS INVERSE IS ALSO POSITIVE DEFINITE.

PROOF: i) WANT TO SHOW THAT GIVEN A S.T. $\underline{x} \cdot A \cdot \underline{x} > 0 \ \forall \underline{x} \in \mathbb{R} \setminus \{\underline{0}\}$ THAT $\exists A^{-1}$. SUPPOSE $\nexists A^{-1}$, IN THIS CASE $\exists \underline{x} \neq \underline{0}$ S.T. $A \underline{x} = \underline{0}$ $\Rightarrow$ $\exists \underline{x} \neq \underline{0}$ S.T. $\underline{x} \cdot A \underline{x} = 0$, CONTRADICTION! SO $\exists A^{-1}$.

ii) WANT TO SHOW A^{-1} IS POSITIVE DEFINITE. SET $\underline{y} = A^{-1} \underline{x}$ SO THAT $A \underline{y} = \underline{x}$ WITH $\underline{x} \neq \underline{0}$ SO THAT $\underline{y} \neq \underline{0}$. WE HAVE $\underline{x} \cdot A^{-1} \underline{x} = \underline{x} \cdot \underline{y} = A \underline{y} \cdot \underline{y} = \underline{y} \cdot A \cdot \underline{y}$. $\underline{y} \cdot A \cdot \underline{y} > 0$ SINCE A IS POSITIVE DEFINITE AND $\underline{y} \neq \underline{0}$ $\Rightarrow$ $\underline{x} \cdot A^{-1} \underline{x} > 0$ SO A^{-1} IS POSITIVE DEFINITE.

6) INVERTIBLE POSITIVE SEMIDEFINITE MATRICES$^{(H)}$

STATEMENT: A POSITIVE SEMIDEFINITE MATRIX IS INVERTIBLE $\Leftrightarrow$ IT IS POSITIVE DEFINITE

PROOF: "IF" SINCE A IS POSITIVE DEFINITE i.e. $\underline{x} \cdot A \cdot \underline{x} > 0 \ \forall \underline{x} \in \mathbb{R}^n \setminus \{\underline{0}\}$ IT IMPLIES A IS POSITIVE SEMIDEFINITE $\underline{x} \cdot A \cdot \underline{x} > 0 \Rightarrow \underline{x} \cdot A \cdot \underline{x} \geq 0 \ \forall \underline{x} \in \mathbb{R}^n \setminus \{\underline{0}\}$ AND BY THE THEOREM OF POSITIVE DEFINITE MATRICES AND INVERTIBILITY WE KNOW A IS INVERTIBLE.

"ONLY IF" LET $\underline{x} \in \mathbb{R}^n$ BE SUCH THAT $\underline{x} \cdot A \cdot \underline{x} = 0$ WHICH BY THEOREM IS EQUIVALENT TO THE HOMOGENEUS LINEAR SYSTEM $A \underline{x} = \underline{0}$.
SINCE A IS INVERTIBLE THE ONLY SOLUTION TO THE SYSTEM IS $\underline{x} = \underline{0}$ $\Rightarrow$ A IS POSITIVE DEFINITE.

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7) SIGN OF A MATRIX AND ITS EIGENVALUES */(H)

STATEMENT: CONSIDER A SYMMETRIC MATRIX A OF ORDER n,
i) A IS POSITIVE DEFINITE $\Longleftrightarrow$ ALL ITS EIGENVALUES > 0
ii) A IS POSITIVE SEMI DEFINITE $\Longleftrightarrow$ ALL ITS EIGENVALUES ≥ 0

PROOF: "ONLY IF" BY HYPOTHESIS $\underline{x} \cdot A \cdot \underline{x} > 0 \ \forall \underline{x} \in \mathbb{R}^n \setminus \{\underline{0}\}$. TAKE $\lambda \in \sigma(A)$ AND $\underline{v} \in \mathbb{R}^n$ CORRESPONDING TO λ SUCH THAT $\|\underline{v}\| = 1$. SINCE $\underline{x} \cdot A \cdot \underline{x} > 0 \ \forall \underline{x} \in \mathbb{R}^n \setminus \{\underline{0}\}$ AND $A\underline{x} = \lambda \underline{x}$ WE HAVE: $\underline{v} A \underline{v} = \underline{v} \lambda \underline{v} = \lambda (\underline{v} \cdot \underline{v}) = \lambda (\|\underline{v}\|^2) = \lambda$ AND SINCE $\underline{v} A \underline{v} > 0 \Longrightarrow \lambda_i > 0 \ \forall i = 1, 2 \dots n$

"IF" BY HYPOTHESIS $\lambda > 0 \ \forall \lambda \in \sigma(A)$, USING SPECTRAL DECOMPOSITION, $A = B \Lambda B^T$. NOW SET $\underline{y} = B^T \cdot \underline{x}$, $\underline{x} \neq \underline{0}$ SUCH THAT $\underline{y}^T = \underline{x}^T \cdot B$. WE HAVE THAT $\underline{x} A \underline{x} = \underline{x}^T A \underline{x} = \underline{x}^T \cdot (B \cdot \Lambda B^T) \underline{x} = \underline{y}^T \cdot \Lambda \underline{y}$. SINCE Λ IS A DIAGONAL MATRIX WITH $\lambda_i > 0$ AS DIAGONAL ENTRIES WE HAVE $\underline{y}^T \Lambda \underline{y} = \sum_{i=1}^{n} \lambda_i \cdot \underline{y}_i^2$ WHICH IS > 0. SO SINCE $\underline{x} A \underline{x} = \underline{y}^T \Lambda \underline{y} \Longrightarrow \underline{x} A \underline{x} > 0 \ \forall \underline{x} \in \mathbb{R}^n \setminus \{\underline{0}\}$

8) SIGN OF A MATRIX AND DIAGONAL ENTRIES (H)

STATEMENT: CONSIDER A SYMMETRIC MATRIX A OF ORDER n,
i) IF A IS POSITIVE DEFINITE $\Longrightarrow a_{ii} > 0 \ \forall i = 1, 2 \dots n$
ii) IF A IS POSITIVE SEMIDEFINITE $\Longrightarrow a_{ii} \geq 0 \ \forall i = 1, 2 \dots n$

PROOF: CONSIDER THE CANONICAL BASIS OF $\mathbb{R}^n$: $\{\underline{e}_1, \underline{e}_2 \dots \underline{e}_n\}$, IT IS ENOUGH TO OBSERVE THAT $f(\underline{e}_i) = a_{ii} \ \forall i = 1, 2 \dots n$

9) SIGN OF A MATRIX AND PRINCIPAL MINORS*

STATEMENT: CONSIDER A SYMMETRIC MATRIX A OF ORDER n,
i) IF A IS POSITIVE DEFINITE $\Longrightarrow$ ALL ITS PRINCIPAL MINORS > 0
ii) IF A IS POSITIVE SEMI-DEFINITE $\Longrightarrow$ ALL ITS PRINCIPAL MINORS ≥ 0

PROOF: LET A BE POSITIVE DEFINITE AND LET A_{ii} BE THE $(n-1)(n-1)$ MATRIX OBTAINED BY CANCELLING THE iTH ROW AND COLUMN OF A. LET $\underline{x} \in \mathbb{R}^{n-1}$ AND CONSIDER $\tilde{\underline{x}} \in \mathbb{R}^{n}$ S.T. $\tilde{\underline{x}}_{-i} = \underline{x}$. WE HAVE THAT $\underline{x} A_{ii} \underline{x} = \tilde{\underline{x}} A \tilde{\underline{x}}$ WHICH IS >0 SO A POSITIVE DEFINITE $\Longrightarrow$ A_{ii} POSITIVE DEFINITE $\Longrightarrow \det(A_{ii}) > 0$. WE APPLY THE SAME REASONING TO SUBMATRICES OF ORDER $(n-2)$, $(n-3)$ ... OBTAINING THE SAME RESULT $\Longrightarrow$ ALL PRINCIPAL MINOR > 0

10) SYLVESTER JACOBI CRITERION*

STATEMENT: A SYMMETRIC MATRIX A OF ORDER n IS:
i) POSITIVE DEFINITE $\Longleftrightarrow$ ITS LEADING PRINCIPAL MINORS ARE ALL > 0
ii) NEGATIVE DEFINITE $\Longleftrightarrow$ ITS LEADING PRINCIPAL MINORS ARE NOT ZERO AND CHANGE SIGN STARTING WITH A NEGATIVE SIGN
iii) INDEFINITE IF ITS LEADING PRINCIPAL MINORS ARE NOT ZERO AND THE SEQUENCE OF THEIR SIGNS DOES NOT RESPECT i) AND ii)

PROOF: i) "ONLY IF" IS TRUE BECAUSE OF THE SIGN OF A MATRIX AND PRINCIPAL MINORS THEOREM. "IF" FOLLOWS FROM BRIOSCHI'S THEOREM, INDEED LET $\underline{x} \in \mathbb{R}^{n}$, $\underline{x} \neq \underline{0}$ AND LET C BE THE TRIANGULAR MATRIX OF BRIOSCHI. SINCE IT HAS UNITARY DIAGONAL ENTRIES $\det(C) = 1 \neq 0$ SO IT IS INVERTIBLE. $\Longrightarrow z = C \underline{x} \neq \underline{0}$. BY BRIOSCHI $\underline{x} A \underline{x} = \sum_{k=1}^{n} \frac{D_k}{D_{k-1}} \cdot z_k^2$, WE HAVE $D_k > 0$ BY HYP AND $z_k^2 > 0$ AS JUST SHOWN $\Longrightarrow \underline{x} A \underline{x} > 0 \quad \forall \underline{x} \in \mathbb{R}^{n} \setminus \{\underline{0}\}$

ii) IT IS THE DUAL NEGATIVE OF VERSION i) SINCE $\det(-A) = (-1)^{n} \cdot \det(A)$ R A SQUARE MATRIX OF ORDER n.

iii) STRAIGHTFORWARD CONSEQUENCE OF i) AND ii)

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DEFINITIONS

EIGENVALUE/EIGENVECTOR DEFINITION (1231)

LET A BE A SYMMETRIC MATRIX OF ORDER m.
A SCALAR $\lambda \in \mathbb{R}$ IS AN EIGENVALUE OF A, AND A VECTOR $\underline{x} \in \mathbb{R}^m$, $\underline{x} \neq \underline{0}$ IS AN EIGENVECTOR OF A IF TOGETHER THEY SOLVE THE EQUATION

$$A\underline{x} = \lambda \underline{x}$$

CHARACTERISTIC POLYNOMIAL DEFINITION (1238)

LET A BE A SYMMETRIC MATRIX OF ORDER m.
THE CHARACTERISTIC POLYNOMIAL $P_A(\lambda): \mathbb{R} \to \mathbb{R}$ OF A IS DEFINED BY:

$$P_A(\lambda) = \det(A - \lambda I)$$

WHERE $\det(A - \lambda \underline{I}) = 0$ IS THE CHARACTERISTIC EQUATION

SPECTRUM DEFINITION

THE SPECTRUM OF A SYMMETRIC MATRIX $A \in M(m)$ IS THE SET OF ALL EIGENVALUES OF A. WE DENOTE IT WITH: $\sigma(A)$

ALGEBRAIC MULTIPLICITY DEFINITION

LET λ_i BE AN EIGENVALUE OF A SYMMETRIC MATRIX $A \in M(m)$.
WE CALL $m(\lambda_i)$ THE ALGEBRAIC MULTIPLICITY OF λ_i, THE NUMBER OF TIMES λ_i APPEARS AS A ROOT OF THE CHARACTERISTIC POLYNOMIAL $P_A(\lambda)$

ORTHOGONAL MATRIX DEFINITION (1245)

A SQUARE MATRIX B IS ORTHOGONAL IF $B^T \cdot B = I$

MONOMIAL DEFINITION

A FUNCTION $f : \mathbb{R}^m \to \mathbb{R}$ IS CALLED MONOMIAL OF DEGREE m WHEN IT IS OF THE TYPE $f(x_1, x_2 \dots x_m) = c\,(x_1^{\alpha_1}, x_2^{\alpha_2} \dots x_m^{\alpha_m})$, WITH $c \neq 0$ AND $\alpha_1 + \alpha_2 \dots + \alpha_m = m$

FORM DEFINITION (1253)

A FUNCTION $f : \mathbb{R}^m \to \mathbb{R}$ IS CALLED A FORM OF DEGREE m, WHEN IT IS THE SUM OF MONOMIALS OF DEGREE m.

LINEAR FORM DEFINITION

A FORM IS CALLED LINEAR IF IT IS OF DEGREE $m = 1$ i.e.:
$f(x_1, x_2 \dots x_m) = c_1 x_1 + c_2 x_2 + \dots + c_m x_m$ WITH $c \neq 0$

QUADRATIC FORM DEFINITION

A FORM IS CALLED QUADRATIC IF IT IS OF DEGREE $m = 2$ i.e.
$f(x_1, x_2 \dots x_m) = c_1 x_1^2 + c_2 x_2^2 + \dots c_m x_m^2 \dots + b_{12} x_1 x_2, b_{13} x_1 x_2 + \dots =$
$\sum_{k=1}^{m} c_k \cdot x_k^2 + \sum_{1 \leq r \leq k \leq m} b_{rk} \cdot x_r \cdot x_k$ WITH AT LEAST ONE $c_k \neq 0$ OR ONE $b_{rk} \neq 0$.

QUADRATIC FORMS DEFINITION (1258)

LET $f(\underline{x}) = \underline{x}^T \cdot A \underline{x}$ BE A QUADRATIC FORM, THEN, TOGETHER WITH A, THEY ARE

1) POSITIVE DEFINITE IF: $f(\underline{x}) = \underline{x}^T \cdot A \underline{x} > 0 \quad \forall \underline{x} \in \mathbb{R}^m \setminus \{\underline{0}\}$
2) POSITIVE SEMI-DEFINITE IF: $f(\underline{x}) = \underline{x}^T \cdot A \underline{x} \geq 0 \quad \forall \underline{x} \in \mathbb{R}^m \setminus \{\underline{0}\}$
3) NEGATIVE DEFINITE IF: $f(\underline{x}) = \underline{x}^T \cdot A \underline{x} < 0 \quad \forall \underline{x} \in \mathbb{R}^m \setminus \{\underline{0}\}$
4) NEGATIVE SEMI-DEFINITE IF: $f(\underline{x}) = \underline{x}^T \cdot A \underline{x} \leq 0 \quad \forall \underline{x} \in \mathbb{R}^m \setminus \{\underline{0}\}$
5) INDEFINITE IF $f(\underline{x}) = \underline{x}^T \cdot A \underline{x}$ CHANGES SIGN

PRINCIPAL MINOR DEFINITION

A MINOR IS A PRINCIPAL MINOR IF THE SUBMATRIX IS OBTAINED BY CANCELLING $m-k$ ROWS AND COLUMNS OF THE SAME INDEX

LEADING / NW PRINCIPAL MINOR DEFINITION

A MINOR IS A LEADING PRINCIPAL MINOR IF THE SUBMATRIX IS OBTAINED BY CANCELLING THE LAST $m-k$ ROWS AND COLUMNS. DENOTED WITH: $D_k = \det(A_k)$

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THEOREMS

CHARACTERISTIC EQUATION THEOREM (1239)

THE CHARACTERISTIC POLYNOMIAL $P_A : \mathbb{R} \to \mathbb{R}$ OF A SYMMETRIC MATRIX A OF ORDER n, HAS DEGREE n AND IS GIVEN BY:

$$P_A(\lambda) = \lambda^n - tr(A) \cdot \lambda^{n-1} + \dots + (-1)^n \det(A)$$

WHERE ALL THE ROOTS ARE REAL

EIGENSPACE THEOREM

LET A BE A SYMMETRIC MATRIX OF ORDER n., IF $\lambda \in \sigma(A)$, THEN THE SET $W_\lambda = \{ \underline{x} \in \mathbb{R}^n : (A - \lambda I)\underline{x} = \underline{0} \}$ IS A VECTOR SUBSPACE OF $\mathbb{R}^n$, CALLED AN EIGENSPACE

SPAN THEOREM (1241)

LET $S \subset \mathbb{R}^n$ BE A SET OF LINEARLY INDEPENDENT VECTORS. THERE EXISTS A SET OF k ORTHONORMAL VECTORS $\tilde{S} \subset \mathbb{R}^n$ S.T. $\text{SPAN}(S) = \text{SPAN}(\tilde{S})$

ORTHOGONAL MATRIX PROPERTIES (1246)

LET B BE A SQUARE MATRIX, THE FOLLOWING ARE EQUIVALENT:

1) B IS ORTHOGONAL
2) B HAS ORTHONORMAL ROWS
3) B HAS ORTHONORMAL COLUMNS
4) B IS INVERTIBLE AND $B^{-1} = B^T$
5) $\det(B)$ IS ± 1

ORTHOGONALLY DIAGONAZIBLE THEOREM (1250)

A SYMMETRIC MATRIX A IS ORTHOGONALLY DIAGONAZIBLE. i.e. $\exists$ AN ORTHOGONAL MATRIX B SUCH THAT $B^T \cdot A \cdot B = \Lambda$, WHERE Λ IS A DIAGONAL MATRIX, WHOSE ENTRIES ARE THE EIGENVALUES OF A, REPEATED ACCORDING TO THEIR MULTIPLICITY

QUADRATIC FORMS AND SYMMETRIC MATRIX THEOREM (1254)

A FUNCTION $f : \mathbb{R}^n \to \mathbb{R}$ IS A QUADRATIC FORM $\Longleftrightarrow$ $\exists !$ NON ZERO SYMMETRIC MATRIX A OF ORDER n SUCH THAT $f(\underline{x}) = \underline{x} \cdot A \cdot \underline{x}$ $\forall \underline{x} \in \mathbb{R}^n$

SEMIDEFINITE MATRICES AND HOMOGENEUS SYSTEMS THEOREM (1262)

LET A BE A SEMIDEFINITE MATRIX, $\forall \underline{x} \in \mathbb{R}^n$ WE HAVE THAT:

$$\underline{x}^T \cdot A \cdot \underline{x} = \underline{0} \Longleftrightarrow A \underline{x} = \underline{0}$$

MULTIPLICITY AND DIMENSION THEOREM (1240)

LET A BE A SYMMETRIC MATRIX OF ORDER n, THEN $\dim W_\lambda = m(\lambda)$ $\forall \lambda \in \sigma(A)$.

BRIOSCHI'S THEOREM (1268)

LET A BE A SYMMETRIC MATRIX OF ORDER n, WHOSE NW PRINCIPAL MINORS ARE ALL NON ZERO. THEN $\exists$ AN UPPER TRIANGULAR MATRIX $C \in M(n)$ WITH ALL 1 AS ENTRIES ALONG THE MAIN DIAGONAL SUCH THAT $\forall \underline{x} \in \mathbb{R}^n$:

$$\underline{x}^T \cdot A \cdot \underline{x} = \sum_{k=1}^{n} \frac{D_k}{D_{k-1}} \cdot z_k^2 \qquad \text{WHERE } \underline{z} = C \cdot \underline{x}$$

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SEMIDEFINITE AND PRINCIPAL MINORS THEOREM (1272)

A SYMMETRIC MATRIX A OF ORDER n IS

1) POSITIVE SEMIDEFINITE $\iff$ ALL ITS PRINCIPAL MINORS ≥ 0

2) NEGATIVE SEMIDEFINITE $\iff$ ALL ITS PRINCIPAL MINORS HAVE THE SAME SIGN $(-1)^k$ OR ARE NULL

POSITIVE SEMI/DEFINITE AND DETERMINANT THEOREM (1264)

LET A BE A SYMMETRIC MATRIX

i) IF A IS POSITIVE DEFINITE $\det(A) > 0$

ii) IF A IS POSITIVE SEMI-DEFINITE $\det(A) \geq 0$

N-Variable Differential Calculus

Proofs, Definitions and Theorems

(2024-2025)

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Updated: February 20, 2025

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PROOFS

UNCONSTRAINED LOCAL OPTIMIZERS* (1444)

STATEMENT: LET $f: U \to \mathbb{R}$ BE TWICE CONTINUOUSLY DIFFERENTIABLE. LET $\underline{\hat{x}} \in U$ BE A STATIONARY POINT.

i) IF $\underline{\hat{x}}$ IS A LOCAL MAXIMIZER (MINIMIZER) ON U, THEN THE QUADRATIC FORM $h \cdot \nabla^2 f(\underline{\hat{x}}) \cdot h$ IS NEGATIVE (POSITIVE) SEMIDEFINITE

ii) IF THE QUADRATIC FORM $h \cdot \nabla^2 f(\underline{\hat{x}}) \cdot h$ IS NEGATIVE (POSITIVE) DEFINITE THEN $\underline{\hat{x}}$ IS A STRONG LOCAL MAXIMIZER (MINIMIZER).

PROOF: PLEASE REFER TO THE BOOK OR YOUR PROFESSOR VERSION FOR THIS SPECIFIC PROOF. (PROPOSITION 1444)

CONCAVE OPTIMIZATION* (1550)

STATEMENT: LET $f: C \to \mathbb{R}$ BE A CONCAVE FUNCTION DIFFERENTIABLE ON $\text{int}\, C$ AND CONTINUOUS ON C. A POINT $\underline{\hat{x}} \in \text{int}\, C$ IS A GLOBAL MAXIMIZER OF f ON C $\iff \nabla f(\underline{\hat{x}}) = \underline{0}$

PROOF: "ONLY IF": FOLLOWS FROM FERMAT'S THEOREM.
"IF": LET $\underline{\hat{x}} \in \text{int}\, C$ BE SUCH THAT $\nabla f(\underline{\hat{x}}) = \underline{0}$. WE WANT TO SHOW THAT $\underline{\hat{x}}$ IS A GLOBAL MAXIMIZER. SINCE f IS CONCAVE WE HAVE THAT: $f(\underline{y}) \leq f(\underline{\hat{x}}) + \nabla f(\underline{\hat{x}}) \cdot (\underline{y} - \underline{\hat{x}})$ $\forall \underline{y} \in \text{int}\, C$. SINCE f IS CONTINUOUS THE INEQUALITY IS EASILY SEEN TO HOLD $\forall \underline{y} \in C$. SINCE $\nabla f(\underline{\hat{x}}) = \underline{0}$ WE CONCLUDE THAT $f(\underline{y}) \leq f(\underline{\hat{x}})$ $\forall \underline{y} \in C$, AS DESIRED.

DEFINITIONS

SECOND ORDER PARTIAL DERIVATIVES DEFINITION

CONSIDER THE PARTIAL DERIVATIVE FUNCTION $\frac{\partial f}{\partial x_i} : U \rightarrow \mathbb{R}$ AT A POINT $\underline{x} \in U$. IN THIS CASE $\forall\, i,j = 1,2 \dots n$ WE HAVE THE PARTIAL DERIVATIVE $\frac{\partial \left(\frac{\partial f}{\partial x_i}\right)}{\partial x_j}(\underline{x})$ WITH RESPECT TO x_j OF THE PARTIAL DERIVATIVE $\frac{\partial f}{\partial x_i}$.

THESE PARTIAL DERIVATIVES ARE CALLED SECOND-ORDER PARTIAL DERIVATIVES OF f AND ARE DENOTED BY: $\frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x})$ OR BY $f''_{x_i x_j}$.
WHEN $i = j$ WE WRITE $\frac{\partial^2 f}{\partial x_i^2}(\underline{x})$.

JACOBIAN MATRIX DEFINITION

THE JACOBIAN MATRIX $Df(x)$ OF AN OPERATOR $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ AT $\underline{x} \in U$ IS A $m \times n$ MATRIX GIVEN BY:

$$Df(\underline{x}) \text{ OR } Jf(\underline{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\underline{x}) & \frac{\partial f_1}{\partial x_2}(\underline{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\underline{x}) \\ \frac{\partial f_2}{\partial x_1}(\underline{x}) & \frac{\partial f_2}{\partial x_2}(\underline{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\underline{x}) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1}(\underline{x}) & \frac{\partial f_m}{\partial x_2}(\underline{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\underline{x}) \end{bmatrix}$$

HESSIAN MATRIX

GIVEN $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ AND $\underline{x} \in U$, WHERE f IS TWICE DERIVABLE AT $\underline{x}$
WE CALL THE HESSIAN MATRIX:

$$\nabla^2 f(\underline{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\underline{x}) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\underline{x}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\underline{x}) & \frac{\partial^2 f}{\partial x_2^2}(\underline{x}) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\underline{x}) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\underline{x}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\underline{x}) \end{bmatrix}$$

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LEAST SQUARES SOLUTION DEFINITION (1109)

A VECTOR $\underline{x}^* \in \mathbb{R}^n$ IS SAID TO BE A LEAST SQUARES SOLUTION OF A LINEAR SYSTEM OF EQUATION: $\overset{(m\times n)}{A} \cdot \overset{(n\times 1)}{\underline{x}} = \overset{(m\times 1)}{\underline{b}}$ IF IT SOLVES THE OPTIMIZATION PROBLEM: $\min_{x} = \|A\underline{x} - \underline{b}\|^2$ SUB $\underline{x} \in \mathbb{R}^n$.

C^2 FUNCTION DEFINITION

A FUNCTION WITH DOMAIN OF DERIVABILITY EQUAL TO A, IS SAID TO BE $f \in C^2(A)$ IF THE SECOND DERIVATIVES ARE CONTINUOUS ON A.

THEOREMS

SCHWARZ THEOREM (1344)

IF $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ HAS CONTINUOUS SECOND ORDER PARTIAL DERIVATIVES AT $\underline{x} \in U$ THEN:

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\underline{x}) \quad \forall\, i, j = 1, 2 \dots n$$

JACOBIAN AND HESSIAN THEOREM (1357)

THE HESSIAN MATRIX OF A FUNCTION $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ IS THE JACOBIAN MATRIX OF ITS DERIVATIVE OPERATOR $\nabla f: D \to \mathbb{R}^n$.

SECOND ORDER TAYLOR EXPANSION THEOREM (1440)

LET $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ BE TWICE CONTINUOUSLY DIFFERENTIABLE.
THEN AT EACH $\underline{x_0} \in U$ WE HAVE:

$$f(\underline{x}) = f(\underline{x_0}) + \nabla f(\underline{x_0})(\underline{x} - \underline{x_0}) + \frac{1}{2}(\underline{x} - \underline{x_0}) \cdot \nabla^2 f(\underline{x_0}) \cdot (\underline{x} - \underline{x_0}) + o(\| \underline{x} - \underline{x_0} \|^2)$$

$\forall\, \underline{x} \in U$ AS $\underline{x} \to \underline{x_0}$

CONCAVE AND STRICTLY CONCAVE THEOREM (1537)

LET $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ BE DIFFERENTIABLE, THEN f IS CONCAVE:

$$\iff f(\underline{y}) \leq f(\underline{x}) + \nabla f(\underline{x})(\underline{y}-\underline{x}) \quad \forall \underline{x}, \underline{y} \in U$$

AND f IS STRICTLY CONCAVE:

$$\iff f(\underline{y}) < f(\underline{x}) + \nabla f(\underline{x})(\underline{y}-\underline{x}) \quad \forall \underline{x}, \underline{y} \in U, \ \underline{x} \neq \underline{y}$$

CONCAVITY AND HESSIAN MATRIX THEOREM (1539)

LET $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ BE TWICE CONTINUOUSLY DIFFERENTIABLE, THEN:

i) f IS CONCAVE $\iff \nabla^2 f(\underline{x})$ IS NEGATIVE SEMIDEFINITE $\forall \underline{x} \in U$

ii) $\nabla^2 f(\underline{x})$ IS NEGATIVE DEFINITE $\forall \underline{x} \in U \implies f$ IS STRICTLY CONCAVE

IMPLICIT FUNCTION EXISTANCE THEOREM (1648)

LET $g: C \to D$, WITH $A \times B \subseteq C$ AND $k \in D$. IF BOTH:

- g IS CONTINUOUS IN y
- $\inf_{y \in B} g(x,y) < k < \sup_{y \in B} g(x,y) \quad \forall x \in A$

THEN $\exists f: A \to B$ S.T. $g(x, f(x)) = k \quad \forall x \in A$

LEAST SQUARES UNIQUE SOLUTION THEOREM (1110)

LET $m \geq n$, THEN THE OPTIMIZATION PROBLEM: $\min_{\underline{x}} = \|A\underline{x} - \underline{b}\|$ SUB $\underline{x} \in \mathbb{R}^n$. HAS A UNIQUE SOLUTION IF $\rho(A) = n$

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LEAST SQUARES AND SUPERCOERCIVITY/CONCAVITY (1111)

CONSIDER A FUNCTION $g:\mathbb{R}^n \to \mathbb{R}$ DEFINED BY $-\|A\underline{x}-\underline{b}\|^2$, IF $\rho(A) = n$ WE HAVE THAT g IS SUPERCOERCIVE AND STRICTLY CONCAVE

UNIQUENESS OF THE IMPLICIT FUNCTION THEOREM (1649)

LET $g: C \to D$, WITH $A \times B \subseteq C$ AND $k \in D$. IF:

- g IS STRICTLY MONOTONE IN y,

THEN $\exists$ AT MOST ONE FUNCTION
$f: A \to B$ IN B^A SUCH THAT $g(x, f(x)) = k$ $\forall x \in A$

IMPLICIT FUNCTION DIFFERENTIABILITY THEOREM (1653)

LET $g: C \subseteq \mathbb{R}^2 \to D$, WITH $A \times B \subseteq C$ AND $k \in D$, SUPPOSE THAT:

i) THE SETS A AND B ARE OPEN
ii) g IS CONTINUOUSLY DIFFERENTIABLE ON $A \times B$, WITH EITHER:
  - $\partial g(x,y)/\partial y > 0$ $\forall (x,y) \in A \times B$ OR
  - $\partial g(x,y)/\partial y < 0$ $\forall (x,y) \in A \times B$

IF $f: A \to B$ IS SUCH THAT $g(x, f(x)) = k$ $\forall x \in A$, THEN

- f IS CONTINUOUSLY DIFFERENTIABLE
- $$f'(x) = -\frac{\frac{\partial g}{\partial x}(x,y)}{\frac{\partial g}{\partial y}(x,y)} \quad \forall (x,y) \in g^{-1}(k) \cap (A \times B)$$

IMPLICIT FUNCTION THEOREM ON $g(x_0, y_0) = 0$ (DINI) (1656)

LET $g: U \rightarrow \mathbb{R}$ BE DEFINED ON AN OPEN SET U OF $\mathbb{R}^2$ AND LET $g(x_0, y_0) = 0$. IF BOTH:

i) g IS CONTINUOUSLY DIFFERENTIABLE ON A NEIGHBORHOOD OF (x_0, y_0)

ii) $\frac{\partial g}{\partial y}(x_0, y_0) \neq 0$

THEN $\exists$ NEIGHBORHOODS $B(x_0)$ AND $V(y_0)$ AND A UNIQUE FUNCTION $f: B(x_0) \rightarrow V(y_0)$ SUCH THAT: $g(x, f(x)) = 0 \quad \forall x \in B(x_0)$

THE FUNCTION f IS CONTINUOUSLY DIFFERENTIABLE ON $B(x_0)$ WITH:

$$f'(x) = -\frac{\frac{\partial g}{\partial x}(x,y)}{\frac{\partial g}{\partial y}(x,y)} \qquad \forall (x,y) \in g^{-1}(0) \cap \left(B(x_0) \times V(y_0)\right)$$

IMPLICIT FUNCTION THEOREM FOR A GENERIC SCALAR k (DINI) (1660)

LET $g: U \rightarrow \mathbb{R}$ BE DEFINED ON AN OPEN SET U OF $\mathbb{R}^2$ AND LET $g(x_0, y_0) = k$ IF BOTH:

i) g IS CONTINUOUSLY DIFFERENTIABLE ON A NEIGHBORHOOD OF (x_0, y_0)

ii) $\frac{\partial g}{\partial y}(x_0, y_0) \neq 0$

THEN $\exists$ NEIGHBORHOODS $B(x_0)$ AND $V(y_0)$ AND A UNIQUE FUNCTION $f: B(x_0) \rightarrow V(y_0)$ SUCH THAT: $g(x, f(x)) = k \quad \forall x \in B(x_0)$

THE FUNCTION f IS CONTINUOUSLY DIFFERENTIABLE ON $B(x_0)$ WITH:

$$f'(x) = -\frac{\frac{\partial g}{\partial x}(x,y)}{\frac{\partial g}{\partial y}(x,y)} \qquad \forall (x,y) \in g^{-1}(k) \cap \left(B(x_0) \times V(y_0)\right)$$

ECONOMICS APPLICATIONS WITH COBB-DOUGLAS.

- PLEASE REFER TO THE BOOK FOR THE ECONOMIC APPLICATIONS:

- (1661)
- (1662)
- (1663)

NECESSARY CONDITION LEMMA 1780 (1780)

LET $\hat{\underline{x}} \in C \cap D$ BE A LOCAL SOLUTION TO THE OPTIMIZATION PROBLEM: $\max f(\underline{x})$ SUB $g(\underline{x}) = b$.
IF $\nabla g(\hat{\underline{x}}) \neq \underline{0}$ THEN $\exists$ A SCALAR $\hat{\lambda} \in \mathbb{R}$ SUCH THAT:

$$\nabla f(\hat{\underline{x}}) = \hat{\lambda} \nabla g(\hat{\underline{x}})$$

LAGRANGE THEOREM (1782)

LET $\hat{x} \in C \cap D$ BE A LOCAL SOLUTION OF THE OPTIMIZATION PROBLEM: $\max f(\underline{x})$ SUB $g(\underline{x}) = b$.
IF $\nabla g(\hat{\underline{x}}) \neq \underline{0}$, THEN $\exists$ A SCALAR $\hat{\lambda} \in \mathbb{R}$, CALLED LAGRANGE MULTIPLIER, SUCH THAT THE PAIR $(\hat{\underline{x}}, \hat{\lambda}) \in \mathbb{R}^{m+1}$ IS A STATIONARY POINT OF THE LAGRANGIAN FUNCTION.

Integral Calculus

Proofs, Definitions and Theorems

(2024-2025)

These notes have been compiled by Luca Penouel as an additional resource to aid BIEM and BIEF students of the **Mathematics 2 course (30063)**.

You will find the proofs, definitions, and theorems listed in the official syllabus for the academic year 2024-2025. The versions of the proofs/definitions/theorems contained herein may differ from those found in the official course textbook; they are adaptations that combine elements from multiple sources to enhance clarity and understanding, while (hopefully) maintaining mathematical correctness.

Please note that this handout **IS NOT** intended to replace the official course materials. It is provided merely as a supplementary tool to assist in your study and understanding of the course content.

The notes are not in its final form; updates and corrections will be applied as needed. If you notice any errors, kindly contact me privately so that I can make the necessary revisions.

For the most current version of these notes, always access the document via the shared drive link. Avoid downloading the PDF directly, as updates will be posted periodically and downloading may result in using outdated material.

Updated: March 6, 2025

* = IN CLASS
H = HOMEWORK

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PROOFS

13) RELATIONSHIP BETWEEN LOWER AND UPPER INTEGRAL *

(1913)

STATEMENT: IF $f: [a,b] \to \mathbb{R}_+$ IS A BOUNDED FUNCTION, THEN BOTH THE LOWER AND THE UPPER INTEGRAL EXIST AND ARE FINITE, WITH

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$$

PROOF: SINCE f IS POSITIVE AND BOUNDED, $\exists\, M > 0$ SUCH THAT $0 \le f(x) \le M \quad \forall x \in [a,b]$. $\implies$ $\forall$ SUBDIVISION $\pi = \{x_i\}_{i=0}^{n}$ WE HAVE:

$$0 \le \inf_{x \in [x_{i-1}, x_i]} f(x) \le \sup_{x \in [x_{i-1}, x_i]} f(x) \le M \qquad \forall i = 1, 2 \dots n$$ WHICH BECOMES:

$$0 \le I(f,\pi) \le S(f,\pi) \le M(b-a) \qquad \forall \pi \in \Pi$$

SO $I(f,\pi)$ AND $S(f,\pi)$ ARE BOUNDED AND BY THE LEAST UPPER BOUND PRINCIPLE $\exists$ SUP OF $I(f,\pi)$ AND INF OF $S(f,\pi)$. i.e. $\exists \underline{\int_a^b} f(x)\,dx, \overline{\int_a^b} f(x)\,dx \in \mathbb{R}_+$

WE PROVED <u>EXISTANCE</u>, NOW WE PROVE <u>INEQUALITY</u>.

SUPPOSE BY CONTRADICTION: $\underline{\int_a^b} f(x)\,dx - \overline{\int_a^b} f(x)\,dx = \varepsilon > 0$

WE KNOW THAT $\exists$ A SUBDIVISION π' SUCH THAT: $I(f,\pi') > \underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2}$

AND A SUBDIVISION π'' SUCH THAT: $S(f,\pi'') < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2}$

THESE TWO YIELD: $I(f,\pi') - S(f,\pi'') > \underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2} - \left(\overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2} \right) = \varepsilon - \varepsilon = 0$

IF WE TAKE THE SUBDIVISION $\pi = \pi' \cup \pi''$, THEN $I(f,\pi) \ge I(f,\pi')$ AND $S(f,\pi) \le S(f,\pi'')$, WE CONCLUDE THAT:

$$I(f,\pi) - S(f,\pi) \ge I(f,\pi') - S(f,\pi'') > 0$$

THAT IS $I(f,\pi) > S(f,\pi)$, WHICH IS A CONTRADICTION!

14) CONTINUOUS FUNCTIONS ARE INTEGRABLE* (1931)

STATEMENT: EVERY CONTINUOUS FUNCTION $f : [a,b] \to \mathbb{R}$ IS INTEGRABLE

PROOF: SINCE f IS CONTINUOUS ON $[a,b]$, BY WEIERSTRASS f IS BOUNDED AND BY HEINE - CANTOR f IS UNIFORMLY CONTINUOUS ON $[a,b]$. i.e. (1) $\exists\ \delta_\varepsilon > 0$ S.T. $|x-y| < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall x,y \in [a,b]$

LET $\pi = \{x_i\}_{i=0}^{m}$ BE A SUBDIVISION OF $[a,b]$ S.T. $|\pi| < \delta_\varepsilon$.
THEN $\forall i = 1,2 \ldots m$ BY (1) WE HAVE: $\max_{x\in[x_{i-1},x_i]} f(x) - \min_{x\in[x_{i-1},x_i]} f(x) < \varepsilon$, WHERE MAX AND MIN EXIST BY WEIERSTRASS.

THEN: 1) $\max_{x\in[x_{i-1},x_i]} f(x) - \min_{x\in[x_{i-1},x_i]} f(x) < \varepsilon$

2) $M_k - m_k < \varepsilon$

3) $\sum_{k=1}^{m} M_k \Delta x_k - \sum_{k=1}^{m} m_k \Delta x_k < \sum_{k=1}^{m} \varepsilon \Delta x_k$

4) $U(f,\pi) - L(f,\pi) < \varepsilon (b-a)$

5) f IS INTEGRABLE.

15) INTEGRABILITY OF PRODUCTS^{H} (1936)

STATEMENT: IF $f,g:[a,b]\to\mathbb{R}$ ARE INTEGRABLE FUNCTIONS, THEN THEIR PRODUCT $f\cdot g:[a,b]\to\mathbb{R}$ IS INTEGRABLE.

PROOF: $f\cdot g$ CAN BE REWRITTEN AS: $fg=\frac{1}{4}\left[(f+g)^2-(f-g)^2\right]$.

BY THE LINEARITY OF THE INTEGRAL, $f+g$ AND $f-g$ ARE INTEGRABLE. SO BOTH $(f+g)^2$ AND $(f-g)^2$ ARE INTEGRABLE THEY ARE CONTINUOUS TRASFORMATIONS OF $(f+g)$ AND $(f-g)$.

NOW BY APPLYING THE LINEARITY OF THE INTEGRAL WE HAVE THAT $(f+g)^2-(f-g)^2$ IS ALSO INTEGRABLE $\Rightarrow$ $f\cdot g$ IS INTEGRABLE.

16) MONOTONICITY* (1938)

STATEMENT: LET $f,g:[a,b]\to\mathbb{R}$ BE TWO INTEGRABLE FUNCTIONS, IF $f\leq g$, THEN $\int_a^b f(x)\,dx\leq\int_a^b g(x)\,dx$

PROOF: SINCE $f\leq g$, IT FOLLOWS THAT $I(f,\pi)\leq I(g,\pi)$ FOR ALL $\pi\in\Pi$.

SO, SINCE f AND g ARE INTEGRABLE $\Rightarrow$ $\int_a^b f(x)\,dx\leq\int_a^b g(x)\,dx$

17) INTEGRALS AND ABSOLUTE VALUE ^{H} (1939)

STATEMENT: LET $f:[a,b] \to \mathbb{R}$ BE AN INTEGRABLE FUNCTION, THEN

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

PROOF: SINCE $f \leq |f|$ AND $-f \leq |f|$ BY THE MONOTONICITY OF THE INTEGRAL IT FOLLOWS THAT $\int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$.

AND BY THE MONOTONICITY AND LINEARITY IT FOLLOWS THAT:

$$-\int_a^b f(x)\,dx = \int_a^b -f(x)\,dx \leq \int_a^b |f(x)|\,dx . \Rightarrow \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

18) SANDWICH / BOUNDEDNESS ^{H} (1940)

STATEMENT: LET $f:[a,b] \to \mathbb{R}$ BE AN INTEGRABLE FUNCTION, THEN BY SETTING $m = \inf_{[a,b]} f(x)$ AND $M = \sup_{[a,b]} f(x)$ WE HAVE:

$$m \cdot (b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

PROOF: WE HAVE $m \leq f(x) \leq M$, $\forall x \in [a,b]$, AND BY MONOTONICITY $\Rightarrow \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx \ \forall x \in [a,b]$.

SINCE $\int_a^b m\,dx = m(b-a)$ AND $\int_a^b M\,dx = M(b-a)$, IT HOLDS.

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19) INTEGRAL MEAN VALUE* (1941)

STATEMENT: LET $f:[a,b]\to\mathbb{R}$ BE A BOUNDED AND INTEGRABLE FUNCTION. THEN SETTING $m=\inf_{[a,b]} f(x)$ AND $M=\sup_{[a,b]} f(x)$, $\exists$ A SCALAR $\lambda\in[m,M]$ SUCH THAT:

$$\int_a^b f(x)\,dx=\lambda(b-a)$$

IN PARTICULAR IF f IS CONTINUOUS, $\exists\ c\in[a,b]$ SUCH THAT $f(c)=\lambda$, SO:

$$\int_a^b f(x)\,dx=f(c)(b-a)$$

PROOF: WE KNOW THAT $m(b-a)\leq\int_a^b f(x)\,dx\leq M(b-a)$ WHICH IMPLIES $m\leq\frac{\int_a^b f(x)\,dx}{b-a}\leq M$. AND LET $\lambda=\frac{\int_a^b f(x)\,dx}{b-a}$.

NOW, SINCE f IS CONTINUOUS BY DARBOUX f ASSUMES ALL THE VALUES INCLUDED BETWEEN m AND M. $\Longrightarrow\ \exists\ c\in[a,b]$ S.T. $f(c)=\lambda$

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20) 1^{ST} FUNDAMENTAL THEOREM OF CALCULUS* (1950)

STATEMENT: LET $P:[a,b] \to \mathbb{R}$ BE A PRIMITIVE FUNCTION OF $f:[a,b] \to \mathbb{R}$. IF f IS RIEMANN INTEGRABLE, THEN:

$$\int_a^b f(x)\,dx = P(b) - P(a)$$

PROOF: LET $\pi = \{x_i\}_{i=0}^{m}$ BE A SUBDIVISION OF $[a,b]$, IF WE ADD AND SUBTRACT $P(x_i)$ $\forall i = 1,2 \dots m$ WE HAVE: $P(b) - P(a) = \sum_{i=1}^{m} \left(P(x_i) - P(x_{i-1})\right)$

SINCE P IS A PRIMITIVE OF f, P IS DIFFERENTIABLE ON (x_{i-1}, x_i), AND CONTINUOUS ON $[x_{i-1}, x_i]$ SO BY LAGRANGE $\forall i = 1,2 \dots m$ $\exists\, \hat{x}_i \in (x_{i-1}, x_i)$ SUCH THAT:

$$P'(\hat{x}_i) = \frac{P(x_i) - P(x_{i-1})}{x_i - x_{i-1}}$$

SINCE P IS A PRIMITIVE WE HAVE:

$$f(\hat{x}_i) = P'(\hat{x}_i) = \frac{P(x_i) - P(x_{i-1})}{x_i - x_{i-1}}$$

SO $P(b) - P(a) = \sum_{i=1}^{m} \left(P(x_i) - P(x_{i-1})\right) = \sum_{i=1}^{m} f(\hat{x}_i) \cdot (x_i - x_{i-1}) = \sum_{i=1}^{m} f(\hat{x}_i) \cdot \Delta x_i$

WHICH IMPLIES: $\underset{\pi \in \Pi}{I}(f, \pi) \le P(b) - P(a) \le \underset{\pi \in \Pi}{S}(f, \pi)$

WHICH HOLDS $\forall \pi \in \Pi$ SO $\Rightarrow \sup_{\pi \in \Pi} I(f, \pi) \le P(b) - P(a) \le \inf_{\pi \in \Pi} S(f, \pi)$

AND SINCE f IS INTEGRABLE $\Rightarrow \int_a^b f(x)\,dx = P(b) - P(a)$

21) 2^{ND} FUNDAMENTAL THEOREM OF CALCULUS* (1955)

* THIS VERSION OF THE PROOF IS VALID AND WAS TAKEN FROM PROFESSOR FEIN NOTES, FOR THE SYLLABUS VERSION, CONSULT THE BOOK.

STATEMEN: LET f BE CONTINUOUS ON $[a,b]$, THEN THE INTEGRAL FUNCTION $\int_{x_0}^{x} f(t)\,dt$ IS CONTINUOUSLY DERIVABLE AND $F'(x) = f(x)$

PROOF: f CONTINUOUS ON $[a,b] \Rightarrow f \in R([a,b])$.

BY THE MEAN VALUE THEOREM OF INTEGRAL CALCULUS WE HAVE: THAT $\forall\, [x, x+h] \subseteq [a,b] \quad \exists\, c \in (x, x+h)$ S.T.

$$f(c) = \frac{1}{h} \cdot \int_{x}^{x+h} f(t)\,dt = \frac{F(x+h) - F(x)}{h}$$

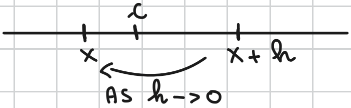
NOW WE PASS TO THE LIMIT AS $h \to 0$, WE HAVE $c \to x$.

AND: $\lim_{h \to 0} f(c) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$ WHICH BECOMES

$$f(x) = F'(x)$$

EXTRA*:

IMAGE TO UNDERSTAND:



22) INTEGRABLE FUNCTIONS ARE LIPSCHITZ* (1954)

STATEMENT: THE INTEGRAL FUNCTION $F:[a,b]\to\mathbb{R}$ OF AN INTEGRABLE FUNCTION $f:[a,b]\to\mathbb{R}$ IS LIPSCHITZ CONTINUOUS

PROOF: SINCE f IS BOUNDED, $\exists\, M>0$ S.T. $|f(x)|\leq M \quad \forall x\in[a,b]$.
WE SHOW $|F(x)-F(y)|\leq M|x-y| \ \forall x,y\in[a,b]$.
CONSIDER $x,y\in[a,b]$ WITH $x>y$. BY DEF. OF INTEGRAL FUNCTION, WE HAVE:

$$|F(x)-F(y)|=\left|\int_y^x f(t)\,dt\right|\leq\int_y^x|f(t)|\,dt\leq\int_y^x M\,dt=M\,|x-y|$$

SO $|F(x)-F(y)|\leq M|x-y|$.

23) LINEARITY OF THE INTEGRAL* (1959)

STATEMENT: LET $f,g:I\to\mathbb{R}$ BE TWO FUNCTIONS THAT ADMIT A PRIMITIVE. THEN $\forall\,\alpha,\beta\in\mathbb{R}$ THE FUNCTION $\alpha f+\beta g:I\to\mathbb{R}$ ADMITS A PRIMITIVE AND: $\int(\alpha f+\beta g)(x)\,dx=\alpha\int f(x)\,dx+\beta\int g(x)\,dx+k$, WITH $k\in\mathbb{R}$

PROOF: SET $P_f=\int f(x)\,dx$ AND $P_g=\int g(x)\,dx$. SINCE $f,g:I\to\mathbb{R}$ ADMIT A PRIMITIVE BOTH P_f AND P_g ARE DEFINED.

TAKE $\alpha P_f+\beta P_g$, $\forall\,\alpha,\beta\in\mathbb{R}$, WHICH IS DERIVABLE AND:

$(\alpha P_f+\beta P_g)'=\alpha f+\beta g$, i.e. $(\alpha P_f+\beta P_g)$ IS THE ANTIDERIVATIVE OF $\alpha f+\beta g$. SO:

$$\int(\alpha f+\beta g)(x)\,dx=\alpha P_f(x)+\beta P_g(x)+c=\alpha\int f(x)\,dx+\beta\int g(x)\,dx$$

24) INTEGRATION BY PARTS* (1961)

STATEMENT: LET $f, g : I \to \mathbb{R}$ BE TWO DIFFERENTIABLE FUNCTIONS. THEN, FOR SOME $k \in \mathbb{R}$ WE HAVE:

$$\int f'(x)\, g(x)\, dx + \int f(x)\, g'(x)\, dx = f(x) \cdot g(x) + k$$

PROOF: BY THE PRODUT RULE $(fg)' = f'g + fg'$. SO $fg = P\, f'g + fg'$,

SO: $f(x) g(x) = \int \left[f'(x) \cdot g(x) + f(x) \cdot g'(x) \right] dx = \int f'(x) g(x)\, dx + \int f(x) g'(x)\, dx + \hat{k}$

FOR SOME $k \in \mathbb{R}$. BY REARRANGING AND SETTING $k = -\hat{k}$ WE ARE DONE.

25) INTEGRATION BY SUBSTITUTION* (1964)

* THIS VERSION OF THE PROOF IS VALID AND WAS TAKEN FROM PROFESSOR FEIN NOTES, FOR THE SYLLABUS VERSION, CONSULT THE BOOK.

STATEMENT: CONSIDER $f : [a,b] \to \mathbb{R}$ DERIVABLE WITH $f' \in R([a,b])$. IF $g : I \subseteq \mathbb{R} \to \mathbb{R}$ IS CONTINUOUS ON THE INTERVAL I WITH $f([a,b]) \subseteq I$ THEN $(g \circ f) \cdot f' \in R([a,b])$ WITH:

$$\int_a^b g(f(x))\, f'(x)\, dx = \int_{f(a)}^{f(b)} g(y)\, dy$$

PROOF: SINCE g IS CONTINUOUS ON I, g ADMITS AN ANTIDERIVATIVE G AND (1): $\int_{f(a)}^{f(b)} g(y)\, dy = G(f(b)) - G(f(a))$.

WE KNOW G IS DERIVABLE SINCE IT IS AN ANTIDERIVATIVE AND ALSO f IS DERIVABLE BY HYPOTHESIS.

SO: $[G(f(x))]' = G'(f(x)) \cdot f'(x) = g(f(x)) \cdot f'(x)$. WHICH MEANS THAT:

(2): $G(f(x))$ IS AN ANTIDERIVATIVE OF $g(f(x)) \cdot f'(x)$.

SO: $\int_a^b g(f(x)) \cdot f'(x)\, dx \overset{(2)}{=} G(f(b)) - G(f(a)) \overset{(1)}{=} \int_{f(a)}^{f(b)} g(y)\, dy$

26) COMPARISON CRITERION* (1988)

STATEMENT: LET $f, g : [a, +\infty) \to \mathbb{R}$ BE TWO POSITIVE FUNCTIONS INTEGRABLE ON EVERY $[a, b] \subseteq [a, +\infty)$ WITH $f \le g$, THEN:

$$\int_a^{+\infty} g(x)\,dx \in [0, \infty) \implies \int_a^{+\infty} f(x)\,dx \in [0, \infty)$$ AND:

$$\int_a^{+\infty} f(x)\,dx = +\infty \implies \int_a^{+\infty} g(x)\,dx = +\infty$$

PROOF: BY THE MONOTONICITY OF THE IMPROPER INTEGRAL THEOREM WE HAVE $\int_a^{+\infty} f(x)\,dx \le \int_a^{+\infty} g(x)\,dx$, AND WE ALSO HAVE $\int_a^{+\infty} f(x)\,dx \in [0, \infty)$ AND $\int_a^{+\infty} g(x)\,dx \in [0, \infty)$.

THEREFORE $\int_a^{+\infty} f(x)\,dx$ CONVERGES IF $\int_a^{+\infty} g(x)\,dx$ CONVERGES WHILE $\int_a^{+\infty} g(x)\,dx$ DIVERGES POSITIVELY IF $\int_a^{+\infty} f(x)\,dx$ DIVERGES POSITIVELY.

27) ASYMPTOTIC COMPARISON CRITERION^{H} (1989)

STATEMENT: LET $f : [a, +\infty) \to \mathbb{R}$ BE A POSITIVE FUNCTION INTEGRABLE ON EVERY INTERVAL $[a, b] \subseteq [a, +\infty)$.

i) IF $f \sim g$ AS $x \to +\infty$, THEN $\int_a^{+\infty} g(x)\,dx$ CONVERGES (DIVERGES POSITIVELY) $\iff \int_a^{+\infty} f(x)\,dx$ CONVERGES (DIVERGES POSITIVELY)

ii) IF $f = o(g)$ AS $x \to +\infty$ AND $\int_a^{+\infty} g(x)\,dx$ CONVERGES, THEN $\int_a^{+\infty} f(x)\,dx$ CONVERGES TOO.

iii) IF $f = o(g)$ AS $x \to +\infty$ AND $\int_a^{+\infty} f(x)\,dx$ DIVERGES POSITIVELY, THEN, $\int_a^{+\infty} g(x)\,dx$ DIVERGES POSITIVELY, TOO.

PROOF: NOT PROVIDED IN THE BOOK.

28) ABSOLUTE CONVERGENCE CRITERION H (1993)

STATEMENT: LET $f:[a,+\infty)\to\mathbb{R}$ BE A FUNCTION INTEGRABLE ON EVERY INTERVAL $[a,b]\subseteq[a,+\infty)$. THE IMPROPER INTEGRAL $\int_a^{+\infty} f(x)\,dx$ CONVERGES IF IT CONVERGES ABSOLUTELY, IN THIS CASE:

$$\left|\int_a^{+\infty} f(x)\,dx\right| \leq \int_a^{+\infty} |f(x)|\,dx$$

PROOF: NOT PROVIDED IN THE BOOK.

29) BARROW-TORRICELLI * (2086)

STATEMENT: A FUNCTION $g:[a,b]\to\mathbb{R}$, WITH $g(a)=0$ IS CONTINUOUSLY DIFFERENTIABLE $\Leftrightarrow \exists!$ CONTINUOUS FUNCTION $\gamma:[a,b]\to\mathbb{R}$ SUCH THAT: $g(x)=\int_a^x \gamma(t)\,dt \quad \forall x\in[a,b]$

THE BIJECTIVE FUNCTION $T: C^1_0([a,b])\to C([a,b])$ THAT TO EACH $g\in C^1_0([a,b])$ ASSOCIATES $\gamma\in C([a,b])$ IS THE DIFFERENTIAL OPERATOR $T(g)=g'$

ITS INVERSE FUNCTION $T^{-1}: C([a,b])\to C^1_0([a,b])$, WHICH TO EACH $\gamma\in C([a,b])$ ASSOCIATES $T^{-1}(\gamma)\in C^1_0([a,b])$, IS THE INTEGRAL OPERATOR

$$T^{-1}(\gamma)(x)=\int_a^x \gamma(t)\,dt \qquad \forall x\in[a,b]$$

PROOF: REFER TO THE BOOK/YOUR PROFESSOR VERSION

DEFINITIONS

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SUBDIVISION / PARTITION DEFINITION (1909)

A SET $\pi = \{x_i\}_{i=0}^{n}$ OF POINTS IS A SUBDIVISION / PARTITION OF AN INTERVAL $[a,b]$, IF: $a = x_0 < x_1 \ldots < x_n = b$

THE SET OF ALL POSSIBLE SUBDIVISIONS OF AN INTERVAL $[a,b]$ IS DENOTED BY Π

LOWER INTEGRAL SUM DEFINITION

LET $a,b \in \mathbb{R}$ WITH $a < b$, LET π BE A SUBDIVISION OF THE INTERVAL $[a,b]$ AND SUPPOSE $f : [a,b] \to \mathbb{R}$ IS BOUNDED.

THE LOWER SUM OF f OVER π IS THE NUMBER $I(f,\pi) = \sum_{k=1}^{n} m_k \cdot \Delta x_k$

UPPER INTEGRAL SUM DEFINITION

LET $a,b \in \mathbb{R}$ WITH $a < b$, LET π BE A SUBDIVISION OF THE INTERVAL $[a,b]$ AND SUPPOSE $f : [a,b] \to \mathbb{R}$ IS BOUNDED.

THE UPPER SUM OF f OVER π IS THE NUMBER $S(f,\pi) = \sum_{k=1}^{n} M_k \cdot \Delta x_k$

REFINEMENT DEFINITION (1910)

GIVEN TWO SUBDIVISIONS π AND π' OF $[a,b]$, WE SAY THAT π' REFINES π IF $\pi \subseteq \pi'$. THAT IS, IF ALL POINTS OF π ARE ALSO POINTS OF π'.

MESH DEFINITION (1917)

GIVEN A SUBDIVISION π OF $[a,b]$, WE DEFINE THE MESH OF π, DENOTED BY $|\pi|$, THE POSITIVE QUANTITY

$$|\pi| = \max_{i=1,2\ldots n} \Delta x_i$$

LOWER AND UPPER INTEGRAL DEFINITION (1912)

LET $f : [a,b] \to \mathbb{R}_+$ BE A BOUNDED FUNCTION.

THE VALUE: $\underline{\int}_a^b f(x)\,dx = \sup_{\pi \in \Pi} I(f,\pi)$ IS SAID TO BE THE LOWER INTEGRAL OF f ON $[a,b]$.

THE VALUE: $\overline{\int}_a^b f(x)\,dx = \inf_{\pi \in \Pi} S(f,\pi)$ IS SAID TO BE THE UPPER INTEGRAL OF f ON $[a,b]$.

RIEMANN INTEGRABLE DEFINITION (1914)

A BOUNDED FUNCTION $f : [a,b] \to \mathbb{R}_+$ IS SAID TO BE RIEMANN INTEGRABLE IF:

$$\underline{\int}_a^b f(x)\,dx = \overline{\int}_a^b f(x)\,dx$$

THIS COMMON VALUE, DENOTED $\int_a^b f(x)\,dx$, IS CALLED THE RIEMANN INTEGRAL OF f ON $[a,b]$.

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POSITIVE/NEGATIVE PARTS OF A f DEFINITION (1918)

LET $f: A \subseteq \mathbb{R} \to \mathbb{R}$, THE FUNCTION $f^+: A \subseteq \mathbb{R} \to \mathbb{R}_+$ IS DEFINED BY:

$f^+(x) = \max\{f(x), 0\}$ $\forall x \in A$, CALLED THE POSITIVE PART

THE FUNCTION $f^-: A \subseteq \mathbb{R} \to \mathbb{R}_+$ IS DEFINED BY:

$f^-(x) = -\min\{f(x), 0\}$ $\forall x \in A$, CALLED THE NEGATIVE PART.

RIEMANN INTEGRABLE DEFINITION FOR FUNCTIONS THAT CHANGE SIGN (1920)

A BOUNDED FUNCTION $f: [a,b] \to \mathbb{R}$ IS SAID TO BE INTEGRABLE IN THE SENSE OF RIEMANN IF THE FUNCTIONS f^+ AND f^- ARE BOTH INTEGRABLE. IN THIS CASE THE RIEMANN INTEGRABLE OF f ON $[a,b]$ IS DEFINED BY:

$$\int_a^b f(x)\,dx = \int_a^b f^+(x)\,dx - \int_a^b f^-(x)\,dx$$

STEP FUNCTION DEFINITION (1928)

A FUNCTION $f: [a,b] \to \mathbb{R}$ IS CALLED A STEP FUNCTION IF $\exists$ A SUBDIVISION $\pi = \{x_i\}_{i=0}^{m}$ AND A SET $\{c\}_{i=1}^{m}$ OF CONSTANTS S.T.

$$f(x) = c_i \qquad \forall x \in (x_{i-1}, x_i)$$

PRIMITIVE / INDEFINITE INTEGRAL DEFINITION (1944)

A FUNCTION $P: I \to \mathbb{R}$ IS CALLED A PRIMITIVE OR INDEFINITE INTEGRAL OF $f: I \to \mathbb{R}$ IF IT IS DIFFERENTIABLE ON I AND:

$$P'(x) = f(x) \quad \forall x \in I$$

WE DENOTE A PRIMITIVE OF f BY $\int f(x)\,dx$

INTEGRAL FUNCTION DEFINITION (1953)

LET $f: [a,b] \to \mathbb{R}$ BE AN INTEGRABLE FUNCTION, THE FUNCTION $F: [a,b] \to \mathbb{R}$ GIVEN BY:

$$F(x) = \int_a^x f(t)\,dt \quad \forall x \in [a,b]$$

IS CALLED THE INTEGRAL FUNCTION OF f.

IMPROPER RIEMANN INTEGRABLE DEFINITION (1969)

LET $f: [a, +\infty) \to \mathbb{R}$ BE A FUNCTION INTEGRABLE ON EVERY INTERVAL $[a,b] \subseteq [a, +\infty)$ WITH INTEGRAL FUNCTION F. IF $\lim_{x \to +\infty} F(x) \in \overline{\mathbb{R}}$, WE SET:

$$\int_a^{+\infty} f(x)\,dx = \lim_{x \to +\infty} F(x)$$

AND THE FUNCTION f IS SAID TO BE INTEGRABLE IN THE IMPROPER SENSE ON $[a, +\infty)$. THE VALUE $\int_a^{+\infty} f(x)\,dx$ IS CALLED THE IMPROPER RIEMANN INTEGRAL.

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IMPROPER RIEMANN INTEGRABLE ON $\mathbb{R}$ DEFINITION (1969)

LET $f:\mathbb{R}\to\mathbb{R}$ BE A FUNCTION INTEGRABLE ON EVERY COMPACT INTERVAL IF $\exists$ THE INTEGRALS $\int_a^{+\infty} f(x)\,dx$ AND $\int_{-\infty}^{a} f(x)\,dx$, THE FUNCTION f IS SAID TO BE INTEGRABLE ON $\mathbb{R}$ AND WE SET:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_a^{+\infty} f(x)\,dx + \int_{-\infty}^{a} f(x)\,dx$$

PROVIDED WE DO NOT HAVE AN INDETERMINATE FORM $+\infty - \infty$.
THE VALUE $\int_{-\infty}^{+\infty} f(x)\,dx$ IS CALLED THE IMPROPER RIEMANN INTEGRAL OF f ON $\mathbb{R}$.

CAUCHY PRINCIPAL VALUE DEFINITION (1980)

LET $f:\mathbb{R}\to\mathbb{R}$ BE A FUNCTION INTEGRABLE ON EVERY INTERVAL $[a,b]$. THE CAUCHY PRINCIPAL VALUE, DENOTED BY $PV\int_{-\infty}^{+\infty} f(x)\,dx$, OF THE NTEGRAL $\int_{-\infty}^{+\infty} f(x)\,dx$ IS GIVEN BY:

$$PV\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{k\to+\infty} \int_{-k}^{k} f(x)\,dx$$

WHENEVER THE LIMIT EXISTS IN $\overline{\mathbb{R}}$

IMPROPER ABSOLUTELY CONVERGENT INTEGRAL DEF. (1992)

LET $f: [a,+\infty) \to \mathbb{R}$ BE A FUNCTION INTEGRABLE ON EVERY INTERVAL $[a,b] \subseteq [a+\infty)$. THE IMPROPER INTEGRAL $\int_a^{+\infty} f(x)\,dx$ CONVERGES IF IT CONVERGES ABSOLUTELY. IN THIS CASE:

$$\left|\int_a^{+\infty} f(x)\,dx\right| \leq \int_a^{+\infty} |f(x)|\,dx$$

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STIELTJES INTEGRAL DEFINITION (2009)

A BOUNDED FUNCTION $f:[a,b]\to\mathbb{R}$ IS SAID TO BE STIELTJES INTEGRABLE WITH RESPECT TO AN INCREASING FUNCTION $g:[a,b]\to\mathbb{R}$ IF:

$$\sup_{\pi\in\Pi} I(f,g,\pi) = \inf_{\pi\in\Pi} S(f,g,\pi)$$

THE COMMON VALUE, DENOTED BY $\int_a^b f(x)\,dg(x)$, IS CALLED THE STIELTJES INTEGRAL OF f WITH RESPECT TO g ON $[a,b]$

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THEOREMS

DERICHLET FUNCTION THEOREM (1916)

LET $f: [a,b] \rightarrow \mathbb{R}$ BE THE DERICHLET FUNCTION:

$$f(x) = \begin{cases} 1 & \text{IF } x \in Q \cap [a,b] \\ 0 & \text{IF } x \in (\mathbb{R} - Q) \cap [a,b] \end{cases}$$

$f(x)$ IS NOT INTEGRABLE IN THE SENSE OF RIEMANN.

INTEGRABLE FUNCTION THEOREM (1921)

A BOUNDED FUNCTION $f: [a,b] \rightarrow \mathbb{R}$ IS INTEGRABLE $\Longleftrightarrow$ $\int_{-a}^{b} f(x)\,dx =$ AND $\int_{a}^{-b} f(x)\,dx$, IN THIS CASE:

$$\int_a^b f(x)\,dx = \int_{-a}^{b} f(x)\,dx = \int_{a}^{-b} f(x)\,dx$$

RIEMANN INTEGRABLE AND SUBDIVISION THEOREM (1925)

A BOUNDED FUNCTION $f: [a,b] \rightarrow \mathbb{R}$ IS RIEMANN INTEGRABLE $\Longleftrightarrow$ $\forall \varepsilon > 0$, $\exists$ A SUBDIVISION π SUCH THAT: $S(f,\pi) - I(f,\pi) < \varepsilon$.

STABILITY OF THE INTEGRAL THEOREM (1926)

LET $f: [a,b]$ BE AN INTEGRABLE FUNCTION. IF $g: [a,b] \rightarrow \mathbb{R}$ IS EQUAL TO f EXCEPT AT MOST A FINITE NUMBER OF POINTS, THEN g IS INTEGRABLE AND $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$

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INTEGRATION PRESERVES CONTINUOUS TRANSFORMATIONS THEOREM (1927)

LET $f: [a,b]$ BE AN INTEGRABLE FUNCTION WITH $m \leq f \leq M$. IF $g: [m, M] \to \mathbb{R}$ IS CONTINUOUS, THEN THE COMPOSITE FUNCTION $g \circ f: [a,b] \to \mathbb{R}$ IS INTEGRABLE

STEP FUNCTIONS ARE RIEMANN INTEGRABLE (1929)

A STEP FUNCTION $f: [a,b] \to \mathbb{R}$, DETERMINED BY THE SUBDIVISION $\{x_i\}_{i=0}^{m}$ AND BY THE CONSTANTS $\{c_i\}_{i=1}^{m}$ IS RIEMANN INTEGRABLE AND WE HAVE:

$$\int_a^b f(x)\,dx = \sum_{i=1}^{m} c_i \cdot \Delta x_i$$

COUNTABLE DISCONTINUITIES AND INTEGRABILITY THEOREM (1932)

EVERY BOUNDED FUNCTION $f: [a,b] \to \mathbb{R}$ WITH AT MOST COUNTABLY MANY DISCONTINUITIES IS INTEGRABLE

MONOTONIC FUNCTIONS AND INTEGRABILITY THEOREM (1934)

EVERY MONOTONE FUNCTION $f: [a,b]$ IS INTEGRABLE.

LINEARITY OF THE INTEGRAL THEOREM (1935)

LET $f, g: [a,b] \to \mathbb{R}$ BE TWO INTEGRABLE FUNCTIONS. THEN $\forall\, \alpha, \beta \in \mathbb{R}$, THE FUNCTION $\alpha f + \beta g: [a,b] \to \mathbb{R}$ IS INTEGRABLE, WITH:

$$\int_a^b (\alpha f + \beta g)(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

ADDITIVITY WRT THE INTERVAL OF INTEGRATION THEOREM (1937)

LET $f: [a,b] \to \mathbb{R}$ BE A BOUNDED AND INTEGRABLE FUNCTION. IF $a < c < b$, THEN:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

VICE-VERSA, IF $f_1: [a,c] \to \mathbb{R}$ AND $f_2: [c,b] \to \mathbb{R}$ ARE BOUNDED AND INTEGRABLE THEN $f: [a,b] \to \mathbb{R}$ DEFINED BY: $f(x) = \begin{cases} f_1(x) & \text{IF } x \in [a,c] \\ f_2(x) & \text{IF } x \in (c,b] \end{cases}$ IS ALSO BOUNDED AND INTEGRABLE, WITH:

$$\int_a^b f(x)\,dx = \int_a^c f_1(x)\,dx + \int_c^b f_2(x)\,dx$$

ZERO FUNCTION CASE THEOREM (1942)

LET $f: [a,b] \to \mathbb{R}$ BE A CONTINUOUS AND POSITIVE FUNCTION, IF $\int_a^b f(x)\,dx = 0$, THEN $f = 0$

CONSTANT OF INTEGRATION THEOREM (1947)

LET $f: I \to \mathbb{R}$ AND $P_1: I \to \mathbb{R}$ BE A PRIMITIVE FUNCTION OF f. A FUNCTION $P_2: I \to \mathbb{R}$ IS A PRIMITIVE OF f ON I $\iff$ $\exists$ A CONSTANT $k \in \mathbb{R}$ SUCH THAT:

$$P_2 = P_1 + k$$

INTEGRAL TEST FOR CONVERGENCE THEOREM (1975)

LET $f: [1, +\infty) \to \mathbb{R}$ AND SET $a_n = \int_n^{n+1} f(x)\,dx$ $\forall n \geq 1$. IF THE INTEGRAL $\int_1^{+\infty} f(x)\,dx$ CONVERGES, THEN THE SERIES $\sum_{n=1}^{+\infty} a_n$ CONVERGES, WITH:

$$\sum_{n=1}^{+\infty} a_n = \int_1^{+\infty} f(x)\,dx$$

THE CONVERSE IS TRUE IF $\lim_{x \to +\infty} f(x) = 0$

LINEARITY OF IMPROPER INTEGRALS THEOREM (1983)

LET $f, g : [a, +\infty)$ BE TWO FUNCTIONS INTEGRABLE ON $[a, +\infty)$. THEN $\forall \alpha, \beta \in \mathbb{R}$, THE FUNCTION $\alpha f + \beta g : [a, +\infty) \to \mathbb{R}$ IS INTEGRABLE ON $[a, +\infty)$ AND:

$$\int_a^{+\infty} (\alpha f + \beta g)(x)\,dx = \alpha \int_a^{+\infty} f(x)\,dx + \beta \int_a^{+\infty} g(x)\,dx$$

PROVIDED THE SECOND TERM IS NOT AN INDETERMINATE FORM $+\infty - \infty$

MONOTONICITY OF IMPROPER INTEGRALS THEOREM (1984)

LET $f, g : [a, +\infty)$ BE TWO FUNCTIONS INTEGRABLE ON $[a, +\infty)$.
IF $f \geq g$ THEN $\int_a^{+\infty} f(x)\,dx \geq \int_a g(x)\,dx$

NECESSARY CONDITION FOR CONVERGENCE THEOREM (1985)

LET $f : [a, +\infty) \to \mathbb{R}$ BE A FUNCTION POSITIVE AND INTEGRABLE ON EVERY INTERVAL $[a, b] \subseteq [a, +\infty)$. THEN f IS INTEGRABLE ON $[a, +\infty)$:

$$\int_a^{+\infty} f(t)\,dt = \sup_{x \in [a, +\infty)} F(x)$$

IN PARTICULAR $\int_a^{+\infty} f(t)\,dt$ CONVERGES ONLY IF $\lim_{x \to +\infty} f(x) = 0$ (IF IT EXISTS)

GAUSS INTEGRAL THEOREM (1996)

IT HOLDS: $\int_0^{+\infty} e^{-x^2}\,dx = \dfrac{\sqrt{\pi}}{2}$

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STIELTJES INTEGRABLE THEOREM (2010)

A BOUNDED FUNCTION $f:[a,b]\to\mathbb{R}$ IS STIELTJES INTEGRABLE WITH RESPECT TO $g:[a,b]\to\mathbb{R}$ IF, $\forall\ \varepsilon>0\ \exists$ A SUBDIVISION $\pi\in\Pi$ SUCH THAT:
$S(f,g,\pi)-I(f,g,\pi)<\varepsilon$

EXISTANCE OF THE STIELTJES INTEGRAL THEOREM (2011)

THE INTEGRAL $\int_a^b f\cdot dg$ EXISTS IF AT LEAST ONE OF THESE TWO ARE MET:

i) $f:[a,b]\to\mathbb{R}$ IS CONTINUOUS

ii) $f:[a,b]\to\mathbb{R}$ IS MONOTONE AND $g:[a,b]\to\mathbb{R}$ IS CONTINUOUS

FINITE MANY DISCONTINUITIES AND STIELTJES INTEGRABILITY THEOREM (2012)

EVERY BOUNDED FUNCTION $f:[a,b]\to\mathbb{R}$ WITH FINITELY MANY DISCONTINUITIES IS STIELTJES INTEGRABLE WITH RESPECT TO $g:[a,b]\to\mathbb{R}$, PROVIDED g IS CONTINUOUS AT SUCH POINTS.

STIELTJES AS RIEMANN INTEGRAL THEOREM (2013)

LET $f:[a,b]\to\mathbb{R}$ BE A BOUNDED FUNCTION, $g:[a,b]\to\mathbb{R}$ DIFFERENTIABLE AND g' RIEMANN INTEGRABLE. THEN f IS STIELTJES INTEGRABLE WITH RESPECT TO g $\Longleftrightarrow$ $f\cdot g'$ IS RIEMANN INTEGRABLE, IN THIS CASE WE HAVE:

$$\int_a^b f(x)\,dg(x)=\int_a^b f(x)\,g'(x)\,dx$$

VARIATION OF THE PREVIOUS THEOREM (2014)

LET g BE THE INTEGRAL FUNCTION OF A RIEMANN INTEGRABLE FUNCTION $\gamma : [a,b] \to \mathbb{R}$, THAT IS $g(x) = \int_a^x \gamma \, dt \quad \forall x \in [a,b]$.
IF $f : [a,b] \to \mathbb{R}$ IS CONTINUOUS WE HAVE:

$$\int_a^b f(x)\, dg(x) = \int_a^b f(x)\, \gamma(x)\, dx$$

ITO'S FORMULA THEOREM (2015)

LET $f : [c,d] \to \mathbb{R}$ BE CONTINUOUSLY DIFFERENTIABLE AND $g : [a,b] \to \mathbb{R}$ CONTINUOUS WITH $\operatorname{im} g \subseteq [c,d]$, THEN

$$f(g(x)) - f(g(a)) = \int_a^x f'(g(t)) \cdot dg(t)$$

STIELTJES INTEGRAL OF A STEP FUNCTION THEOREM (2017)

LET $f : [a,b] \to \mathbb{R}$ BE CONTINUOUS AND $g : [a,b] \to \mathbb{R}$ BE A STEP FUNCTION WITH DISCONTINUITIES AT THE POINTS $\{d_1, d_2 \dots d_m\}$ OF THE INTERVAL $[a,b]$. WE HAVE:

$$\int_a^b f \, dg = \sum_{j=1}^{m} f(d_j) \left[g(d_j^+) - g(d_j^-) \right]$$

INTEGRATION BY PARTS FOR STIELTJES INTEGRALS THEOREM (2019)

GIVEN ANY TWO INCREASING AND CONTINUOUS FUNCTIONS $f, g : [a,b] \to \mathbb{R}$ IT HOLDS:

$$\int_a^b f \, dg + \int_a^b g \, df = f(b) \cdot g(b) - f(a) \cdot g(a)$$

FOR DOUBTS OR SUGGESTIONS ON THE HANDOUTS

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