

BIEM/BIEF

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BLAB

HANDOUTS

MATHEMATICS
(MODULE 1)
-FIRST PARTIAL-

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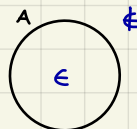
sets

THE ELEMENTS OF THE SET

- \in = belongs
- \notin = doesn't belong
- \emptyset = empty set

HOW TO DESCRIBE A SET

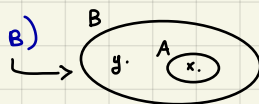
- $A = \{a, b, c\} = \{b, c, a\}$
- $A = \{x\} = \text{singleton}$
- Venn diagram = graphical representation of a set \rightarrow



RELATIONSHIPS BETWEEN SETS

INCLUSION RELATIONSHIP

- $\forall x \in A, x \in B$
- $A \subseteq B$ = contained \rightarrow if $x \in A$ then $x \in B$ (A is a subset of B)
- transitive = $A \subseteq B$ ($B \subseteq C$) $\rightarrow A \subseteq C$



EQUALITY RELATIONSHIP

- $A = B \rightarrow \begin{cases} \text{if } x \in A, \text{ then } x \in B \\ \text{if } x \in B, \text{ then } x \in A \end{cases} \begin{cases} A \subseteq B \\ B \subseteq A \end{cases}$

STRICT INCLUSION RELATIONSHIP

- $A \subset B$ = A strictly contained in B
- $A \subseteq B$ can be $A = B \rightarrow A \subset B$ can't be $A = B$ (by definition)

POWER SETS

- $P(A)$ = set of all subsets of A
- ex. $A = \{1, 2, 3\} \rightarrow P(A) = \{\emptyset, A, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$
- \rightarrow A contains n elements = $P(A)$ contains 2^n elements

SET OPERATIONS

- $A \cap B$ = intersection \rightarrow $x: x \in A$ and $x \in B$
- $A \cup B$ = union \rightarrow $x: x \in A$ or $x \in B$
- $A \setminus B$ = difference \rightarrow $x: x \in A, x \notin B$
- U = universe \rightarrow $A^c = \bar{A} = U - A$ \rightarrow complement \rightarrow $A \setminus B = A \cap B^c$

NUMBER SETS

- natural numbers $\rightarrow \mathcal{N} = \{0, 1, 2, 3, \dots\}$
- integers $\rightarrow \mathbb{Z} = \{0, +1, -1, +2, -2, +3, -3, \dots\}$
- rationals $\rightarrow \mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$
- irrationals $\rightarrow \mathbb{R} = \left\{ \sqrt{2}, \pi, \sqrt{\frac{2}{3}}, \dots \right\} \rightarrow \mathbb{R} - \mathbb{Q}$

\mathbb{Q} is dense, but not continuous while \mathbb{R} is dense and continuous

$$\forall x, y \in \mathbb{Q}$$

$$\exists z \in \mathbb{Q} : x < z < y$$

there is a 1-to-1 correspondence between \mathbb{R} and the real line

STRUCTURES IN \mathbb{R}

- operations = $x+y, x-y, xy, x/y$
- order = $x > y, x = y, x < y, x \geq y, x \leq y \rightarrow$
 - given two real numbers x, y you can always compare them (one and only one relation holds)
 - the order is a total order
 - $x > 0$ = strictly positive, $x < 0$ = strictly negative
 - $x \geq 0$ = positive, $x \leq 0$ = negative

INTERVALS

- $[x, y]$ = endpoints included
- (x, y) = endpoints excluded

- BOUNDED \rightarrow

- $[a, b]$ = closed $\rightarrow \{x : a \leq x \leq b\}$
- (a, b) = open $\rightarrow \{x : a < x < b\}$
- $[a, b)$ = half-closed / open $\rightarrow \{x : a \leq x < b\}$
- $(a, b]$ = half-closed / open $\rightarrow \{x : a < x \leq b\}$

- UNBOUNDED \rightarrow

- $[a, +\infty)$ = closed (unbounded on right/above) $\rightarrow \{x : x \geq a\}$
- $(a, +\infty)$ = open (unbounded on right/above) $\rightarrow \{x : x > a\}$
- $(-\infty, a]$ = closed (unbounded on left/below) $\rightarrow \{x : x \leq a\}$
- $(-\infty, a)$ = open (unbounded on left/below) $\rightarrow \{x : x < a\}$
- $(-\infty, +\infty) = \mathbb{R}$ (total)

DEFINITION OF INTERVAL

$$A \subseteq \mathbb{R}, A \neq \emptyset$$

A is called an interval if: $\forall x, y \in A$, the set $\{z \in \mathbb{R} : x \leq z \leq y\} \subseteq A$

$$\text{EXTENDED REAL LINE} \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$$

UPPER AND LOWER BOUND

$A \subseteq \mathbb{R}, A \neq \emptyset \Rightarrow$ a number $n \in \mathbb{R}$ is called an upper bound of A if

$$n \geq x \quad \forall x \in A$$

\Rightarrow a number $n \in \mathbb{R}$ is called a lower bound of A if

$$n \leq x \quad \forall x \in A$$

- A^* = set of all upper bounds

- A_* = set of all lower bounds

BOUNDED SET

- bounded above ($A \neq \emptyset$) = it has at least one upper bound $\rightarrow \exists n \in \mathbb{R}:$

$$n \geq x \quad \forall x \in A \rightarrow A^* \neq \emptyset$$

- bounded below ($A \neq \emptyset$) = it has at least one lower bound $\rightarrow \exists n \in \mathbb{R}:$

$$n \leq x \quad \forall x \in A \rightarrow A_* \neq \emptyset$$

- bounded ($A \neq \emptyset$) = both bounded above and below

- unbounded ($A \neq \emptyset$) = not bounded

MAXIMUM

$$A \subseteq \mathbb{R}, A \neq \emptyset$$

an element $\hat{x} \in \mathbb{R}$ is called a maximum of A if \rightarrow

$$\textcircled{1} \hat{x} \in A$$

$$\textcircled{2} \hat{x} \geq x \quad \forall x \in A$$

MINIMUM

$$A \subseteq \mathbb{R}, A \neq \emptyset$$

an element $\hat{x} \in \mathbb{R}$ is called a minimum of A if \rightarrow

$$\textcircled{1} \hat{x} \in A$$

$$\textcircled{2} \hat{x} \leq x \quad \forall x \in A$$

SUPREUM

$$A \subseteq \mathbb{R}, A \neq \emptyset$$

we call supremum of A ($\sup A$) the least upper bound of A that is

$$\sup A = \min A^*$$

if A has a supremum it is unique

INFIMUM

$$A \subseteq \mathbb{R}, A \neq \emptyset$$

we call infimum of A ($\inf A$) the greatest lower bound of A that is

$$\inf A = \max A_*$$

if A has an infimum it is unique

└─┘

- if $\max A \exists \rightarrow \max A = \sup A$

- if $\min A \exists \rightarrow \min A = \inf A$

- if $\sup A \exists \rightarrow \exists \max A = \sup A \leftrightarrow \sup A \in A$

- if $\inf A \exists \rightarrow \exists \min A = \inf A \leftrightarrow \inf A \in A$

└─┘

COMPLETE LEST THEOREM

- A bounded above $\Rightarrow \exists \sup A$

- A bounded below $\Rightarrow \exists \inf A$

if $A \subseteq \mathbb{R}$ is unbounded above instead of saying that $\sup A \nexists$ sometimes

we say that $\sup A = +\infty$

if $A \subseteq \mathbb{R}$ is unbounded below instead of saying that $\inf A \nexists$ sometimes

we say that $\inf A = -\infty$

ABSOLUTE VALUE

def. $\rightarrow \forall x \in \mathbb{R}, |x| =$

$$\begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \rightarrow |x| = \text{distance from } x \text{ to } 0$$

DISTANCE

$$\forall x, y \in \mathbb{R}, \text{dist}(x, y) = |y - x|$$

NEIGHBOURHOODS

$$x_0 \in \mathbb{R}$$

we call neighbourhood of x_0 any open bounded interval (a, b) such that x_0 is the midpoint

radius = half-length of the neighbourhood

PROPERTIES

- distributive I = $\forall \alpha, \beta \in \mathbb{R}, \forall \underline{x} \in \mathbb{R}^n, (\alpha + \beta) \underline{x} = \alpha \underline{x} + \beta \underline{x}$
- distributive II = $\forall \alpha \in \mathbb{R}, \forall \underline{x}, \underline{y} \in \mathbb{R}^n, \alpha (\underline{x} + \underline{y}) = \alpha \underline{x} + \alpha \underline{y}$
- associative = $\forall \alpha, \beta \in \mathbb{R}, \forall \underline{x} \in \mathbb{R}^n, (\alpha \beta) \underline{x} = \alpha (\beta \underline{x}) \rightarrow \alpha \beta \underline{x}$
- 1 is neutral for vectors = $\forall \underline{x} \in \mathbb{R}^n, 1 \underline{x} = \underline{x}$

LINEAR COMBINATIONS

$$\forall \alpha, \beta \in \mathbb{R}, \forall \underline{x}, \underline{y} \in \mathbb{R}^n, \alpha \underline{x} + \beta \underline{y} \in \mathbb{R}^n \rightarrow \begin{array}{l} \forall \alpha_1, \dots, \alpha_m \in \mathbb{R} \\ \forall \underline{x}_1, \dots, \underline{x}_m \in \mathbb{R}^n \\ \hookrightarrow \alpha_1 \underline{x}_1 + \dots + \alpha_m \underline{x}_m \in \mathbb{R}^n \end{array}$$

is said to be a linear combination and $\alpha_1, \dots, \alpha_m$ are the coefficients of the linear combination

INNER / SCALAR PRODUCT

$$\underline{x} \cdot \underline{y} = \forall \underline{x} = (x_1, \dots, x_n), \underline{y} = (y_1, \dots, y_n) \in \mathbb{R}^n, \underline{x} \cdot \underline{y} = x_1 y_1 + \dots + x_n y_n \in \mathbb{R}$$

PROPERTIES

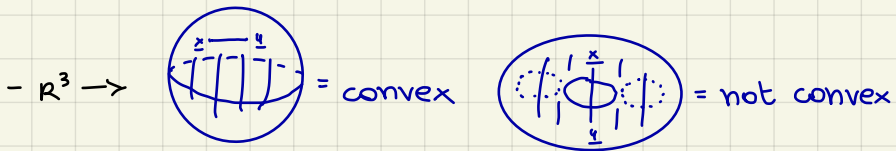
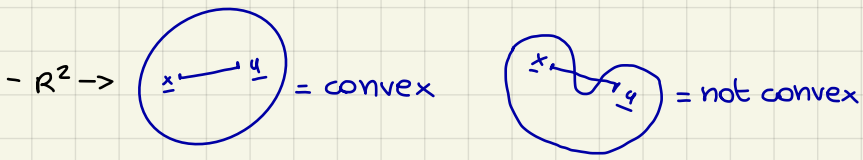
- commutative = $\forall \underline{x}, \underline{y} \in \mathbb{R}^n, \underline{x} \cdot \underline{y} = \underline{y} \cdot \underline{x}$
- distributive = $\forall \underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^n, (\underline{x} \cdot \underline{y}) \cdot \underline{z} = \underline{x} \cdot \underline{z} + \underline{y} \cdot \underline{z}$
- associative = $\forall \alpha \in \mathbb{R}, \forall \underline{x}, \underline{y} \in \mathbb{R}^n, (\alpha \underline{x}) \cdot \underline{y} = \alpha (\underline{x} \cdot \underline{y}) \rightarrow \alpha \cdot \underline{x} \cdot \underline{y}$
- non-negativity = $\forall \underline{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \underline{x} \cdot \underline{x} = x_1^2 + \dots + x_n^2 \geq 0$ and $\underline{x} \cdot \underline{x} = 0 \leftrightarrow \underline{x} = \underline{0}$

ORDER

- $\underline{x} \leq \underline{y}$ (smaller than or equal) = $\forall \underline{x}, \underline{y} \in \mathbb{R}^n, \underline{x} \leq \underline{y}$ if $\begin{cases} x_1 \leq y_1 \\ \vdots \\ x_n \leq y_n \end{cases}$
- $\underline{x} < \underline{y}$ (smaller) = $\forall \underline{x}, \underline{y} \in \mathbb{R}^n, \underline{x} < \underline{y}$ if $\begin{cases} x_1 \leq y_1 \\ \vdots \\ x_n \leq y_n \end{cases}$ and at least one of these ineq. is $<$ (smaller)
- $\underline{x} \ll \underline{y}$ (strictly smaller) = $\forall \underline{x}, \underline{y} \in \mathbb{R}^n, \underline{x} \ll \underline{y}$ if $\begin{cases} x_1 < y_1 \\ \vdots \\ x_n < y_n \end{cases}$

INTERVALS \rightarrow CONVEX SETS

- $\mathbb{R} \rightarrow A \subseteq \mathbb{R}, A \neq \emptyset$ is an interval if: $\forall x, y \in A$, the segment with endpoints x, y is $\subseteq A$



- $\mathbb{R}^n \rightarrow$ how can we define a segment in \mathbb{R}^n , if there is no geometry? we must give a purely algebraic definition

SEGMENT IN \mathbb{R}^n def. = consider $\underline{x}, \underline{y} \in \mathbb{R}^n$ we call segment with endpoints $\underline{x}, \underline{y}$ the set of all $\underline{z} \in \mathbb{R}^n$: $\underline{z} = \alpha \underline{x} + (1 - \alpha) \underline{y}, 0 \leq \alpha \leq 1$

\downarrow
convex linear combinations $\underline{x}, \underline{y}$

also: $\alpha \underline{x} + \beta \underline{y} \rightarrow$

$$0 \leq \alpha \leq 1$$

$$0 \leq \beta \leq 1$$

$$\alpha + \beta = 1$$

CONVEX SETS = $\mathbb{R}^n = A \subseteq \mathbb{R}^n, A \neq \emptyset$ is a convex set if $\forall x, y \in A$ the segment with endpoints $\underline{x}, \underline{y}$ is $\subseteq A$

- \mathbb{R} = intervals
- $\mathbb{R}^2, \mathbb{R}^3$ = "geometric" convex sets
- $n > 3$ = the formal, algebraic, abstract notion of convex sets

NORM

$\mathbb{R}^n \rightarrow \forall \underline{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \|\underline{x}\| = \sqrt{x_1^2 + \dots + x_n^2} = \sqrt{\underline{x} \cdot \underline{x}} \rightarrow$

$n = 2, 3 \rightarrow \|\underline{z}\| = \text{dist}(\underline{z}, \underline{0})$

$n = 1 \rightarrow \underline{z} = x$

$\|x\| = \sqrt{x^2} = |x| = \text{dist}(x, 0)$

PROPERTIES

- non-negativity = $\forall \underline{x} \in \mathbb{R}^n, \|\underline{x}\| \geq 0$ and $\|\underline{x}\| = 0 \iff \underline{x} = \underline{0}$
- $\|\alpha \underline{x}\| = \forall \alpha \in \mathbb{R}, \forall \underline{x} \in \mathbb{R}^n, \|\alpha \underline{x}\| = |\alpha| \|\underline{x}\|$
- Cauchy-Schwarz inequality = $|\underline{x} \cdot \underline{y}| = \forall \underline{x}, \underline{y} \in \mathbb{R}^n, |\underline{x} \cdot \underline{y}| \leq \|\underline{x}\| \|\underline{y}\|$
- triangle inequality = $\|\underline{x} + \underline{y}\| = \forall \underline{x}, \underline{y} \in \mathbb{R}^n, \|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

UNIT VECTOR = $\underline{x} \in \mathbb{R}^n, \|\underline{x}\| = 1 \rightarrow \underline{x} \cdot \underline{x} = 1$

ORTHOGONAL VECTORS = $\underline{x}, \underline{y} \in \mathbb{R}^n, \underline{x} \cdot \underline{y} = 0 \rightarrow \underline{x} \perp \underline{y}$

ORTHONORMAL VECTOR = $\underline{x}, \underline{y} \in \mathbb{R}^n \rightarrow$ orthogonal + unit

DISTANCE \rightarrow EUCLIDEAN DISTANCE

$\mathbb{R} \rightarrow \text{dist}(x, y) = |y - x|$

$\mathbb{R}^n \rightarrow \forall \underline{x} = (x_1, \dots, x_n), \underline{y} = (y_1, \dots, y_n) \in \mathbb{R}^n, \text{dist}(\underline{x}, \underline{y}) = \|\underline{y} - \underline{x}\| = \sqrt{(y_1 - x_1)^2 + \dots + (y_n - x_n)^2}$

PROPERTIES

- symmetry = $\forall \underline{x}, \underline{y} \in \mathbb{R}^n, d(\underline{x}, \underline{y}) = d(\underline{y}, \underline{x})$
- non-negativity = $\forall \underline{x}, \underline{y} \in \mathbb{R}^n, d(\underline{x}, \underline{y}) \geq 0$ and $d(\underline{x}, \underline{y}) = 0 \iff \underline{y} = \underline{x}$
- triangle-inequality = $\forall \underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^n, d(\underline{x}, \underline{z}) \leq d(\underline{x}, \underline{y}) + d(\underline{y}, \underline{z})$

NEIGHBOURHOOD

$\mathbb{R} \rightarrow$ a neighbourhood $B_r(x_0)$ of $x_0 \in \mathbb{R}$ is any open bounded interval such that x_0 is the midpoint = $(x_0 - r, x_0 + r)$

But if we want to generalize it to \mathbb{R}^n , it is better if we first try and transform it into an equivalent definition $\rightarrow \{x \in \mathbb{R}^n : d(x_0, x) < r\}$

$\mathbb{R}^n \rightarrow \forall x_0 \in \mathbb{R}^n$, a neighbourhood of x_0 is any set: $B_r(x_0) = \{\underline{x} \in \mathbb{R}^n : d(\underline{x}, x_0) < r\}$

POSITIONS OF $\underline{x}_0 \in \mathbb{R}^n$ w/r TO $A \subseteq \mathbb{R}^n$

INTERIOR POINT

$A \subseteq \mathbb{R}^n, \underline{x}_0 \in \mathbb{R}^n \rightarrow x_0$ is an interior point for A if \exists a neighbourhood $B(x_0) \subseteq A$

EXTERIOR POINT

$A \subseteq \mathbb{R}^n, x_0 \in \mathbb{R}^n \rightarrow x_0$ is an exterior point for A if \exists a neighbourhood $B(x_0) \subseteq A^c$

BOUNDARY POINT

$A \subseteq \mathbb{R}^n, x_0 \in \mathbb{R}^n \rightarrow x_0$ is a boundary point for A if x_0 is not interior and not exterior that is if $\rightarrow \forall$ neighbourhood $B(x_0), B(x_0) \cap A \neq \emptyset, B(x_0) \cap A^c \neq \emptyset$

CATEGORIES OF SETS $A \subseteq \mathbb{R}^n$

OPEN SETS

$A \subseteq \mathbb{R}^n \rightarrow A$ is open in \mathbb{R}^n if $\forall x_0 \in A \rightarrow x_0$ interior point for A
in \mathbb{R} , every open interval is an open set but there are open sets which are not intervals \rightarrow when we say that A is open we are saying that $A \subseteq \text{int } A$
($A \subseteq \mathbb{R}^n$ is open when $A = \text{int } A \rightarrow A$ contains all its boundary points = A is closed)

CLOSED SETS

$A \subseteq \mathbb{R}^n \rightarrow A$ is closed in \mathbb{R}^n if $\forall x_0$ boundary point for $A \rightarrow x_0 \in A$
in \mathbb{R} , every closed interval is a closed set but there are closed sets which are not intervals \rightarrow when we say that A is closed we are saying that $\bar{A} \subseteq A$ ($A \subseteq \mathbb{R}^n$ is closed when $A = \bar{A} \rightarrow A$ contains none of its boundary points = A is open)



- if A contains only a part of its boundary points then A is neither open nor closed
- only \mathbb{R}^n and \emptyset are both open and closed

BOUNDED / UNBOUNDED SETS

- $A \subseteq \mathbb{R}^n \rightarrow A$ is bounded if \exists a neighbourhood $B(o)$ such that $A \subseteq B(o)$
- if it \nexists then A is unbounded

COMPACT SETS

$A \subseteq \mathbb{R}^n$ is compact if it is closed and bounded in \mathbb{R}^n

ISOLATED AND LIMIT POINTS

ISOLATED POINTS

$A \subseteq \mathbb{R}^n, x_0 \in \mathbb{R}^n \rightarrow x_0$ is an isolated point for A if \rightarrow ① $x_0 \in A$

• x_0 interior \rightarrow cannot be isolated

• x_0 boundary \rightarrow maybe or maybe not isolated

② \exists a neighbourhood

$$B(x_0) \cap A = \{x_0\}$$

LIMIT POINTS

$A \subseteq \mathbb{R}^n, x_0 \in \mathbb{R}^n \rightarrow x_0$ is a limit point for A if $\rightarrow \forall$ neighbourhood $B(x_0)$

$$\exists x, x \neq x_0 : x \in B(x_0) \cap A$$

REMARKS

- x_0 interior $\rightarrow x_0$ limit point
- x_0 boundary \rightarrow maybe or maybe not limit point
- x_0 boundary point, $x_0 \in A \rightarrow x_0$ can be either limit point or isolated point (not both)
- x_0 boundary point, $x_0 \notin A \rightarrow x_0$ must be limit point
- given a set $A \subseteq \mathbb{R}^n$, the set of all the limit points of A is called the derived set denoted by A'
- $A \subseteq \mathbb{R}^n$ is closed if $A' \subseteq A$

theorems and proofs for structures

DOUBLE COMPLEMENT

\forall universal set U , \forall set $A \subseteq U$,
 $(A^c)^c = A$

PROOF

For two sets A, B , the claim that $A = B$ holds equivalently to the claim that both the relations $A \subseteq B$ and $B \subseteq A$ hold by definition. Therefore, proving that $(A^c)^c = A$ holds is equivalent to proving that both $(A^c)^c \subseteq A$ and $A \subseteq (A^c)^c$ hold.

So, $(A^c)^c \subseteq A$.

Suppose we take a general element $x \in (A^c)^c$. By definition of complement set, this means that $x \in U$ and $x \notin A^c$. Using the def. of complement set again, this means that $x \in U$ and $x \in A$, which proves that $(A^c)^c \subseteq A$.

$A \subseteq (A^c)^c$.

Suppose we take a general element $x \in A$. By def. of complement set, this means that $x \in U$ and $x \notin A^c$. Using the def. of complement set again, this means that $x \in U$ and $x \in (A^c)^c$, which proves that $A \subseteq (A^c)^c$.

Therefore, $(A^c)^c = A$.

DE MORGAN'S LAWS

Consider \mathcal{U} universe, $A, B \subseteq \mathcal{U}$

$$[A \cap B]^c = A^c \cup B^c$$

PROOF

We first prove that $[A \cap B]^c \subseteq A^c \cup B^c$.

$$\begin{aligned} \forall x \in [A \cap B]^c &\rightarrow x \notin (A \cap B) \rightarrow x \notin A \text{ or } x \notin B \rightarrow x \in A^c \text{ or } x \in B^c \\ &\rightarrow x \in (A^c \cup B^c) \end{aligned}$$

Then we prove that $A^c \cup B^c \subseteq [A \cap B]^c$.

$$\begin{aligned} \forall x \in (A^c \cup B^c) &\rightarrow x \in A^c \text{ or } x \in B^c \rightarrow x \notin A \text{ or } x \notin B \rightarrow x \notin \\ &[A \cap B] \rightarrow x \in [A \cap B]^c \end{aligned}$$

Therefore, $[A \cap B]^c = A^c \cup B^c$

Consider \mathcal{U} universe, $A, B \subseteq \mathcal{U}$

$$[A \cup B]^c = A^c \cap B^c$$

PROOF

We first prove that $[A \cup B]^c \subseteq A^c \cap B^c$.

$$\begin{aligned} \forall x \in [A \cup B]^c &\rightarrow x \notin (A \cup B) \rightarrow x \notin A \text{ or } x \notin B \rightarrow x \in A^c \text{ or} \\ &x \in B^c \rightarrow x \in (A^c \cap B^c) \end{aligned}$$

Then we prove that $A^c \cap B^c \subseteq [A \cup B]^c$.

$$\begin{aligned} \forall x \in (A^c \cap B^c) &\rightarrow x \in A^c \text{ or } x \in B^c \rightarrow x \notin A \text{ or } x \notin B \rightarrow x \notin (A \cup B) \\ &\rightarrow x \in [A \cup B]^c \end{aligned}$$

Therefore, $[A \cup B]^c = A^c \cap B^c$

UNIQUENESS THEOREM FOR THE MAXIMUM/MINIMUM

$$A \subseteq \mathbb{R}, A \neq \emptyset$$

if A has a maximum, it is unique

if A has a minimum, it is unique

PROOF

Suppose x_1, x_2 are maximum for A , this means that:

$$\textcircled{1} x_1, x_2 \in A$$

$$\textcircled{2} x_1 \geq x \quad \forall x \in A \quad (x_1 \text{ is a maximum})$$

$$\textcircled{3} x_2 \geq x \quad \forall x \in A \quad (x_2 \text{ is a maximum})$$

These conditions imply that

$$x_1 \geq x_2 \text{ and } x_2 \geq x_1 \rightarrow x_1 = x_2$$

if a maximum exists then it is unique.

Suppose x_1, x_2 are minimum for A , this means that:

$$\textcircled{1} x_1, x_2 \in A$$

$$\textcircled{2} x_1 \leq x \quad (x_1 \text{ is a minimum})$$

$$\textcircled{3} x_2 \leq x \quad (x_2 \text{ is a minimum})$$

These conditions imply that

$$x_1 \leq x_2 \text{ and } x_2 \leq x_1 \rightarrow x_1 = x_2$$

if a minimum exists then it is unique

CHARACTERIZATION FOR SUPRENUM IN \mathbb{R} USING LEFT/RIGHT NEIGHBORHOOD

consider $A \subseteq \mathbb{R}$, $a \in \mathbb{R}$, then:

$a = \sup A \iff$ (if and only if)

① $a \geq x \quad \forall x \in A$

② $\forall \varepsilon > 0, \exists x \in A$ such that:

$x \in (a - \varepsilon, a]$

PROOF

if $a = \sup A$, ① is satisfied

let $\varepsilon > 0$

since $\sup A(a) > a - \varepsilon$ the point $a - \varepsilon$ is not an upper bound of A
therefore there exists $x \in A$ such that $x > a - \varepsilon$

TRIANGLE INEQUALITY

$$\forall x, y \in \mathbb{R}^n$$

$$\|x + y\| \leq \|x\| + \|y\|$$

PROOF

Since all terms of the inequality that we have to prove are ≥ 0 , the inequality that we have to prove is equivalent to:

$$\|x + y\|^2 \leq (\|x\| + \|y\|)^2$$

that is:

$$\|x + y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$$

that is:

$$(x + y)(x + y) \leq x \cdot x + y \cdot y + 2\|x\|\|y\|$$

that is:

$$x \cdot y \leq \|x\| \cdot \|y\|$$

but Cauchy-Schwarz inequality for the norm guarantees that:

$$\forall x, y \in \mathbb{R}^n,$$

$$|x \cdot y| \leq \|x\| \cdot \|y\|$$

and since it is $x \cdot y \leq |x \cdot y|$, we immediately get the required inequality:

$$x \cdot y \leq \|x\| \cdot \|y\|$$

THE PYTHAGOREAN THEOREM IN \mathbb{R}^n

given $x, y \in \mathbb{R}^n$

if $x \cdot y = y \cdot x = 0$ then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

PROOF

$$\|x+y\|^2 = \left(\sqrt{(x+y) \cdot (x+y)} \right)^2 \rightarrow$$

$$(x+y) \cdot (x+y) \rightarrow$$

$$\|x\|^2 + x \cdot y + y \cdot x + \|y\|^2 \rightarrow$$

$$\|x\|^2 + \|y\|^2$$

ALL NEIGHBORHOODS ARE OPEN SETS

PROOF

take $x_0 \in \mathbb{R}^n$

take a neighborhood $B_\epsilon(x_0)$ in \mathbb{R}^n with radius ϵ

take any $x \in B_\epsilon(x_0)$: we have to show that x is interior for $B_\epsilon(x_0)$

we know $\text{dist}(x, x_0) = d < \epsilon$

consider any number ϵ' such that:

$$0 < \epsilon' < \epsilon - d$$

now take the neighborhood $B_{\epsilon'}(x)$ with radius ϵ' : we have to know that

$$B_{\epsilon'}(x) \subseteq B_\epsilon(x_0)$$

then:

$$d(y, x_0) \leq d(y, x) + d(x, x_0) < \epsilon' + d < \epsilon$$

so $y \in B_\epsilon(x_0)$

this proves that $B_{\epsilon'}(x) \subseteq B_\epsilon(x_0)$

A SET IS CLOSED IF AND ONLY IF ITS COMPLEMENT SET IS OPEN

PROOF

suppose A is closed, $x \in A^c$, x not limit point of A

\exists a neighbourhood $B(x_0)$ such that $A \cap B(x_0) = \emptyset$, such that $B(x_0) \subset A^c$

$\Rightarrow x \in \text{int } A^c = A^c$ open

suppose A^c is open, x limit point of $A \Rightarrow \forall$ neighbourhood of x contains

a point of $A = x \notin \text{int } A^c \Rightarrow A^c$ not open $= x \in A \Rightarrow A$ closed

A SET IS CLOSED IF AND ONLY IF IT CONTAINS ALL ITS LIMIT POINTS
A closed $\leftrightarrow A' \subseteq A$ (A' = set of all limit points of A)

PROOF

A closed, $A = \bar{A}$

$\bar{A} = A \cup A' \Rightarrow A = \bar{A} = A \cup A', A' \subseteq A$

$A' \subseteq A \Rightarrow A \cup A' = A$ but $A \cup A' = \bar{A}$ then $A = \bar{A} \Rightarrow A$ closed

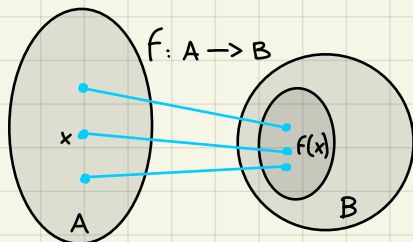
function(s) = $f: A \rightarrow B$

\downarrow \downarrow
 domain codomain

f is a function when for every element of the domain it exists a unique element in B "in relation" with the element of A

$$\forall x \in A \exists ! y \in B \text{ such that } y = f(x)$$

- $y = f(x) \rightarrow y$ is the image of x (inverse image)
- an element $x \in A$ always has exactly one image $y = f(x)$
- range = subset of $f(A)$ of the codomain B (either the image or the codomain of the function)



REAL FUNCTION OF ONE REAL VARIABLE

$$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R} \quad \begin{cases} A \subseteq \mathbb{R} = \text{domain} \\ B = \mathbb{R} = \text{codomain} \end{cases}$$

any law f that associates elements $x \in A$ and elements $y \in B$ such that:

$$\forall x \in A \exists ! y \in \mathbb{R} \text{ such that } y = f(x)$$

- the domain of a function $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a set $A \subseteq \mathbb{R} \rightarrow A = \mathbb{R}$
 $A \subset \mathbb{R}$
 natural domain = the largest subset $A \subseteq \mathbb{R}$ on which that function can be defined

- the codomain of a function $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is always the set \mathbb{R}
- the range of a function $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a set $f(A) \subseteq \mathbb{R} \rightarrow f(A) = \mathbb{R}$
 $f(A) \subset \mathbb{R}$
- a function $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ can always be represented in the cartesian plane xOy with its cartesian graph \rightarrow
 $\text{graph}(f) = \{ P = (x, y) \text{ such that } x \in A, y = f(x) \}$

TYPES OF FUNCTIONS

- $f: A \rightarrow B = \text{surjective} \rightarrow f(A) = B$
 - every element of B it's hit at least once
 - its graph intersects every horizontal line at least once
- $f: A \rightarrow B = \text{injective} \rightarrow \forall x_1, x_2 \in A, x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$
 - no element $y \in B$ is used more than once
 - its graph intersects every horizontal line at most once

③ $f: A \rightarrow B = \text{bijective} \rightarrow$ it is both injective and surjective \rightarrow every $x \in A$ hits only one $y \in B$ and every $y \in B$ is hit (at most once)

MAIN EXAMPLES OF FUNCTIONS

REAL FUNCTIONS OF ONE REAL VARIABLE

$$f: \underbrace{A \subseteq \mathbb{R}}_{\text{domain}} \rightarrow \underbrace{\mathbb{R}}_{\text{codomain}} \quad y = f(x) \rightarrow \text{ex. } y = x^2, y = \sqrt{x}, \dots$$

REAL FUNCTIONS OF n REAL VARIABLES

$$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \quad y = f(\underline{x}) \rightarrow \text{ex. } f: (x_1, x_2, x_3) \in \mathbb{R}^3 \rightarrow x_1^2 + x_2 x_3 \in \mathbb{R}$$

VECTOR FUNCTIONS OF n REAL VARIABLES

$$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m \quad \underline{y} = f(\underline{x}) \rightarrow \text{ex. } f: (x_1, x_2, x_3) \in \mathbb{R}^3 \rightarrow (y_1, y_2) = \begin{pmatrix} x_1^2 + x_3 \sqrt{x_2} \\ x_1 + x_2 \ln(x_3) \end{pmatrix} \in \mathbb{R}^2$$

PIECE WISE DEFINED FUNCTIONS

a function which is constructed with two or more "pieces" that are already known functions

$$\text{ex. } y = \begin{cases} x^2 & x < 0 \\ x+3 & x \geq 0 \end{cases} \quad y = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

COMPOSITION OF FUNCTIONS

Suppose $f: A \rightarrow B$, $g: C \rightarrow D$

if $f(A) \subseteq C$ we can define the composition function $g \circ f: A \rightarrow D$ as:

$$(g \circ f)(x) = g(\underbrace{f(x)}) \quad \forall x \in A$$

$g \circ f$ means that we first apply f and then g

OPERATIONS WITH FUNCTIONS

- sum $\rightarrow f(x) + g(x)$
- product $\rightarrow f(x) \cdot g(x)$

INVERSE FUNCTION

$f: A \rightarrow B$ we say f is invertible on A if there exists a function

$f^{-1}: f(A) \rightarrow A$ (not necessarily defined on all B) which is said to be

inverse function of f such that $\rightarrow \begin{cases} f^{-1}(f(x)) = x & \forall x \in A \\ f(f^{-1}(x)) = x & \forall x \in f(A) \end{cases}$

\hookrightarrow needs to be injective

$$\left. \begin{array}{l} f^{-1}(f(x)) = x \quad \forall x \in A \\ f(f^{-1}(x)) = x \quad \forall x \in f(A) \end{array} \right\}$$

PROPERTIES OF $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$

① f BOUNDED

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is bounded if its range $f(A)$ is a bounded set in $\mathbb{R} \rightarrow$ otherwise it is UNBOUNDED

- bounded below = if its range $f(A)$ is bounded only below in \mathbb{R}
- bounded above = if its range $f(A)$ is bounded only above in \mathbb{R}

② f INCREASING

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called increasing if $\forall x_1, x_2 \in A, x_1 > x_2 \rightarrow f(x_1) \geq f(x_2)$

f STRICTLY INCREASING

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called strictly increasing if $\forall x_1, x_2 \in A, x_1 > x_2 \rightarrow f(x_1) > f(x_2)$

③ f DECREASING

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called decreasing if $\forall x_1, x_2 \in A, x_1 > x_2 \rightarrow f(x_1) \leq f(x_2)$

f STRICTLY DECREASING

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called strictly decreasing if $\forall x_1, x_2 \in A, x_1 > x_2 \rightarrow f(x_1) < f(x_2)$

④ f MONOTONE

if $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is increasing on all A or if it is decreasing on all $A \rightarrow$ monotone on A

f STRICTLY MONOTONE

if $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing on all A or if it is strictly decreasing on all $A \rightarrow$ strictly monotone on A

GLOBAL MAXIMUM

take $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}, x_0 \in A$

we say that x_0 is a global maximizer (or a point of global maximum) for f if: $f(x_0) \geq f(x) \forall x \in A \rightarrow f(x_0)$ is called a global maximum

- there can be many global maximizers ($\max_{x \in A} f(x)$)
- there can either be none or one global maximum ($\arg \max_{x \in A} f(x) = \{x_0 \in A : f(x_0) = \max_{x \in A} f(x)\}$)

STRONG GLOBAL MAXIMUM

take $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}, x_0 \in A$

we say that x_0 is a strong global maximizer if $f(x_0) > f(x) \forall x \in A, x \neq x_0$

$f(x_0)$ is also called a strong global maximum

SUPRENUM

suppose a function f is bounded or at least bounded above
by definition it means that the range $f(A)$ is bounded above
this does not imply that the function f has a global maximum
it implies that it has a supremum $\rightarrow \sup f = \sup$ of the range of f

GLOBAL MINIMUM

take $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $x_0 \in A$

we say that x_0 is a global minimizer (or a point of global minimum)

- or f if: $f(x_0) \leq f(x) \quad \forall x \in A \rightarrow f(x_0)$ is called a global minimum

- there can be many global minimizers ($\min f(x) \quad x \in A$)

- there can either be none or one global maximum

($\arg \min f(x) \quad x \in A = \{x_0 \in A : f(x_0) = \min f(x) \quad x \in A\}$)

STRONG GLOBAL MINIMUM

take $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $x_0 \in A$

we say that x_0 is a strong global minimizer if $f(x_0) < f(x) \quad \forall x \in A$

$x \neq x_0$

$f(x_0)$ is also called a strong global minimum

INFIMUM

suppose a function f is bounded or at least bounded below

by definition it means that the range $f(A)$ is bounded below

this does not imply that the function f has a global minimum

it implies that it has a minimum $\rightarrow \inf f = \inf$ of the range of f

LOCAL MAXIMUM

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $x_0 \in A$

x_0 is a local maximizer (or a point of local maximum) for f if \exists a

neighbourhood $B(x_0)$ such that: $f(x_0) \geq f(x) \quad \forall x \in B(x_0) \cap A \rightarrow f(x_0)$ is

called a local maximum

- global maximizer $\xleftrightarrow{\Leftarrow}$ local maximizer

STRONG LOCAL MAXIMUM

take $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $x_0 \in A$

we say that x_0 is a strong local maximizer if $f(x_0) > f(x)$

$\forall x \in B(x_0) \cap A$, $x \neq x_0$

$f(x_0)$ is also called a strong local maximum

LOCAL MINIMIZER

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}, x_0 \in A$

x_0 is a local minimizer (or a point of local minimum) for f if \exists a neighborhood $B(x_0)$ such that: $f(x_0) \leq f(x) \forall x \in B(x_0) \cap A \rightarrow f(x_0)$ is called a local minimum

- global minimizer $\xleftrightarrow{\leftarrow}$ local minimizer

STRONG LOCAL MINIMIZER

take $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}, x_0 \in A$

we say that x_0 is a strong local minimizer if $f(x_0) < f(x)$

$\forall x \in B(x_0) \cap A, x \neq x_0$

$f(x_0)$ is also called a strong local minimum

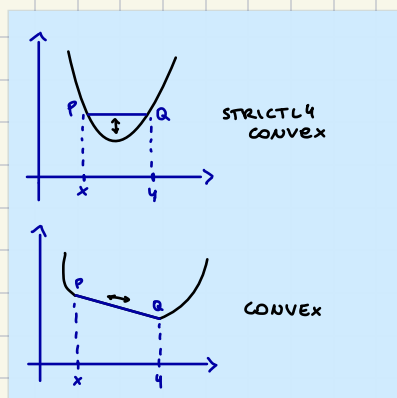
CONCAVE AND CONVEX FUNCTIONS

CONVEX $\rightarrow \cup$

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}, A$ interval

① f is strictly convex if: $\forall x, y \in A$ supposing you call P, Q the corresponding points on the curve, the segment \overline{PQ} is higher than the curve

② f is convex if: $\forall x, y \in A$ supposing you call P, Q the corresponding points on the curve, the segment \overline{PQ} is not lower than the curve

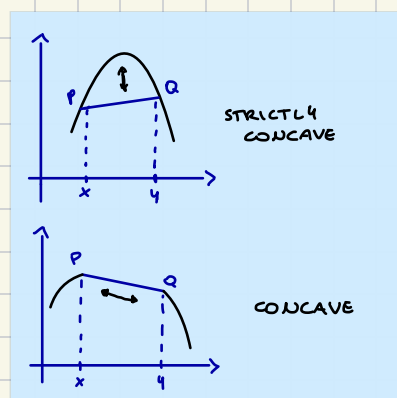


CONCAVE $\rightarrow \cap$

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}, A$ interval

① f is strictly concave if: $\forall x, y \in A$ supposing you call P, Q the corresponding points on the curve, the segment \overline{PQ} is lower than the curve

② f is concave if: $\forall x, y \in A$ supposing you call P, Q the corresponding points on the curve, the segment \overline{PQ} is not lower than the curve



AFFINE = f is convex and concave

STRICTLY CONVEX

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, A interval

f is strictly convex on A if:

$\forall x, y \in A, x \neq y, \forall \alpha: 0 < \alpha < 1, \alpha f(x) + (1-\alpha)f(y) > f[\alpha x + (1-\alpha)y] \rightarrow$
the segment is higher than the curve

CONVEX

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, A interval

f is convex on A if:

$\forall x, y \in A, \forall \alpha: 0 \leq \alpha \leq 1, \alpha f(x) + (1-\alpha)f(y) \geq f[\alpha x + (1-\alpha)y] \rightarrow$
the segment is not higher than the curve

STRICTLY CONCAVE

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, A interval

f is strictly concave on A if:

$\forall x, y \in A, x \neq y, \forall \alpha: 0 > \alpha > 1, \alpha f(x) + (1-\alpha)f(y) < f[\alpha x + (1-\alpha)y] \rightarrow$
the segment is lower than the curve

CONCAVE

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, A interval

f is concave on A if:

$\forall x, y \in A, \forall \alpha: 0 \geq \alpha \geq 1, \alpha f(x) + (1-\alpha)f(y) \leq f[\alpha x + (1-\alpha)y] \rightarrow$
the segment is not lower than the curve

FENCHEL'S THEOREM (LOCAL / GLOBAL)

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, A interval

① if f is convex on A then: x_0 is a local minimizer $\leftrightarrow x_0$ is a global minimizer

② if f is concave on A then: x_0 is a local maximizer $\leftrightarrow x_0$ is a global maximizer

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$

DOMAIN = $A \subseteq \mathbb{R}^n$

CODOMAIN = always \mathbb{R}

RANGE = $f(A) \subseteq \mathbb{R}$

NATURAL DOMAIN = the largest possible domain $A \subseteq \mathbb{R}^n$ on which f can be defined

LEVEL CURVES (instead of 3D graphs) \rightarrow

- given a surface $y = f(x_1, x_2)$ we cut it with horizontal planes $y = k$ and we represent the horizontal slices that we obtain

- given a function $f: A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ we call level curve of height k the curve

$$\{(x_1, x_2) \in A : f(x_1, x_2) = k\} = (f=k) \subseteq A$$

- given a function $y = f(x_1, x_2)$, two different level curves $(f=k_1)$ and $(f=k_2)$ cannot have intersections $\rightarrow (f=k_1) \cap (f=k_2) = \emptyset$
otherwise there would be a point (x_1, x_2) such that $f(x_1, x_2) = k_1$, $f(x_1, x_2) = k_2$, with $k_1 \neq k_2 \rightarrow$ impossible since f is a function

LEVEL SETS

given a function $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ we call level set of height k the set:
 $\{(x_1, \dots, x_n) \in A : f(x_1, x_2, \dots, x_n) = k\} = (f=k) \subseteq A \rightarrow$

- $n=2 \rightarrow$ level curves
- $n > 2 \rightarrow$ difficult/impossible to visualize
- $n=1 \rightarrow f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}, \{x \in A, f(x) = k\} = (f=k) \subseteq A$

EXAMPLES IN ECOLOGICS

① Cobb-Douglas function $\rightarrow y = \sqrt{x_1 x_2} = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$

- natural domain = $x_1 \geq 0, x_2 \geq 0$

- range = $y \geq 0$

- generalizing $\rightarrow y = x_1^{\alpha_1} x_2^{\alpha_2} \rightarrow y = x_1^{\alpha_1} \dots x_n^{\alpha_n}$
 $\hookrightarrow \alpha_1, \alpha_2 > 0; \quad \hookrightarrow \alpha_1, \dots, \alpha_n > 0;$
 $\alpha_1 + \alpha_2 = 1 \quad \alpha_1 + \dots + \alpha_n = 1$

- often used as an example of utility function

② Utility function \rightarrow any function $n(x_1, \dots, x_n)$ defined $\forall x_1, \dots, x_n \geq 0$ such that we can use it to say that the vector $\underline{x} = (x_1, \dots, x_n)$ [a bundle of goods] is preferred to the vector $\underline{y} = (y_1, \dots, y_n)$ [another bundle of goods] $\leftrightarrow n(\underline{x}) \geq n(\underline{y})$ [the utility of \underline{x} is \geq the utility of \underline{y}]

- for an utility function $n(x_1, x_2)$ its level curves are called its indifference curves

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ UN/BOUNDED

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded if its range $f(A) \subseteq \mathbb{R}$ is a bounded set in \mathbb{R} otherwise it's unbounded

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ GLOBAL MAXIMA/MINIMA

GLOBAL MINIMA

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, \underline{x}_0 \in A$

\underline{x}_0 is a global minimizer for f if $f(\underline{x}_0) \leq f(\underline{x}) \quad \forall \underline{x} \in A \rightarrow f(\underline{x}_0)$ is

called global minimum

STRONG MINIMUM

a point of global minimum x_0 is strong if $f(x_0) < f(x) \forall x \in A, x_0 \neq x$

GLOBAL MAXIMA

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, x_0 \in A$

x_0 is a global maximizer for f if $f(x_0) \geq f(x) \forall x \in A \rightarrow f(x_0)$ is called global maximum

STRONG MAXIMUM

a point of global maximum x_0 is strong if $f(x_0) > f(x) \forall x \in A, x_0 \neq x$

SUP/INF

for a function $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ you can also define sup and inf \rightarrow they are just the sup and inf of its range $f(A)$ when they exist

LOCAL MAXIMA/MINIMA

LOCAL MAXIMA

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, x_0 \in A$

x_0 is a local maximizer (or point of local maximum) for f if \exists a neighborhood $B(x_0)$ such that $f(x_0) \geq f(x) \forall x \in B(x_0) \cap A \rightarrow f(x_0)$ is called local maximum

LOCAL MINIMA

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, x_0 \in A$

x_0 is a local minimizer (or point of local minimum) for f if \exists a neighborhood $B(x_0)$ such that $f(x_0) \leq f(x) \forall x \in B(x_0) \cap A \rightarrow f(x_0)$ is called local minimum

f CONVEX/CONCAVE

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, A$ convex set in \mathbb{R}^n (generalization of interval)

① f is strictly convex on A if:

$\forall x, y \in A, x \neq y, \forall \alpha: 0 < \alpha < 1$
 $\alpha f(x) + (1-\alpha)f(y) > f[\alpha x + (1-\alpha)y]$ } \rightarrow the segment is higher than the graph of $f(x)$

② f is convex on A if:

$\forall x, y \in A, x \neq y, \forall \alpha: 0 \leq \alpha \leq 1$
 $\alpha f(x) + (1-\alpha)f(y) \geq f[\alpha x + (1-\alpha)y]$ } \rightarrow the segment is not lower than the graph of $f(x)$

- ③ f is strictly concave on A if:
 $\forall x, y \in A, x \neq y, \forall \alpha: 0 > \alpha > 1$
 $\alpha f(x) + (1-\alpha)f(y) < f[\alpha x + (1-\alpha)y]$ } → the segment is lower than the graph of $f(x)$
- ④ f is concave on A if:
 $\forall x, y \in A, x \neq y, \forall \alpha: 0 \geq \alpha \geq 1$
 $\alpha f(x) + (1-\alpha)f(y) \leq f[\alpha x + (1-\alpha)y]$ } → the segment is not higher than the graph of $f(x)$

CONTOUR SETS

$f: C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ ($C =$ convex set)

UPPER CONTOUR SET OF LEVEL k

$$\{(x_1, \dots, x_n) \in C : f(x_1, \dots, x_n) \geq k\} = (f \geq k) \subseteq C$$

LOWER CONTOUR SET OF LEVEL k

$$\{(x_1, \dots, x_n) \in C : f(x_1, \dots, x_n) \leq k\} = (f \leq k) \subseteq C$$

$$\} \rightarrow (f = k) = (f \geq k) \cap (f \leq k)$$

↳

- if $f: C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is concave → all its upper contour sets $(f \geq k)$ are convex
- if $f: C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is convex → all its lower contour sets $(f \leq k)$ are convex
- suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is strictly monotone (strictly increasing or strictly decreasing) then → ① all its upper contour sets are convex
 ② all its lower contour sets are convex

regardless of the fact that f is concave, convex or neither of them

- all concave functions have convex upper contour sets but the converse is

DIVERSIFICATION PRINCIPLE (UTILITY FUNCTION)

if two bundles of goods $x, y \in C$ ensure a certain level of utility then any convex linear combination $\alpha x + (1-\alpha)y$ of them will give at least the same utility → $u(x) \geq k, u(y) \geq k \rightarrow \forall \alpha: 0 \leq \alpha \leq 1, u[\alpha x + (1-\alpha)y] \geq k \rightarrow$
 all the upper contour sets of the utility function u have to be convex

QUASI-CONCAVITY / QUASI-CONVEXITY

- ① a function $f: C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is quasi-concave if:
 $\forall x, y \in C, \forall \alpha: 0 \leq \alpha \leq 1, f[\alpha x + (1-\alpha)y] \geq \min\{f(x), f(y)\}$
- ② a function $f: C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is quasi-convex if:
 $\forall x, y \in C, \forall \alpha: 0 \leq \alpha \leq 1, f[\alpha x + (1-\alpha)y] \leq \max\{f(x), f(y)\}$
- ③ a function $f: C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is strictly quasi-concave if:
 $\forall x, y \in C, \forall \alpha: 0 < \alpha < 1, f[\alpha x + (1-\alpha)y] > \min\{f(x), f(y)\}$
- ④ a function $f: C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is strictly quasi-convex if:
 $\forall x, y \in C, \forall \alpha: 0 < \alpha < 1, f[\alpha x + (1-\alpha)y] < \max\{f(x), f(y)\}$

↳

take the function $f: C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ then \rightarrow

- f concave $\rightarrow f$ quasi-concave
- f convex $\rightarrow f$ quasi-convex
- f strictly concave $\rightarrow f$ strictly quasi-concave
- f strictly convex $\rightarrow f$ strictly quasi-convex

↳

suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is strictly monotone (strictly increasing or strictly decreasing)

- \rightarrow ① f is strictly quasi-concave therefore also quasi-concave
 - ② f is strictly quasi-convex therefore also quasi-convex
- regardless of the fact that f is concave, convex or neither of them

theorems and proofs for functions

A MONOTONE FUNCTION IS INVERTIBLE IF AND ONLY IF IT IS STRICTLY MONOTONE

an increasing function $f: A \subset \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing if and only if

$$f(x) = f(y) \Rightarrow x = y \quad \forall x, y \in A$$

that is if and only if it is injective

strictly monotone functions are therefore injective and also invertible

PROOF

only if \Rightarrow let f be strictly increasing and assume by contradiction that

$$f(x) = f(y) \text{ for some } x \neq y \text{ and assume } x > y$$

since f strictly increasing $f(x) > f(y)$ should hold however this contradicts the assumption that $f(x) = f(y)$

therefore the assumption $f(x) = f(y)$ for $x \neq y$ is false which implies $x = y$
hence f is injective

if \Rightarrow assume f is increasing but not strictly increasing and we aim to show that $f(x) \neq f(y)$ when $x > y$

let $x > y$ and by increasing monotonicity we know $f(x) \geq f(y)$

if $f(x) = f(y)$ injectivity ($f(x) = f(y) \Rightarrow x = y$) would imply that $x = y$
contradicting the assumption that $x > y$

therefore $f(x) > f(y)$ proving the function is strictly increasing

STRICT MONOTONICITY IS EQUIVALENT TO AN ORDER ISOMORPHISM

a function $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing if and only if $x \geq y \Leftrightarrow f(x) \geq f(y)$

$\forall x, y \in A$

PROOF

to see why we can replace \Rightarrow with \Leftrightarrow it is enough to observe that for a strictly increasing and so increasing function f we have $x \geq y \Rightarrow f(x) \geq f(y)$

$\forall x, y \in A$

A MAXIMIZER IS STRONG IF AND ONLY IF IT IS UNIQUE

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}, C \subseteq A$

$x_0 \in C$ if $f(x_0) > f(x) \forall x \in C, x \neq x_0, x_0 =$ strong maximizer

PROOF

suppose there exist two strong global maximizers: x_1, x_2

$f(x_1) > f(x_2) \Rightarrow f(x_2) > f(x_1) \Rightarrow x_1 = x_2$ (the point is unique)

A MINIMIZER IS STRONG IF AND ONLY IF IT IS UNIQUE

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}, C \subseteq A$

$x_0 \in C$ if $f(x_0) < f(x) \forall x \in C, x \neq x_0, x_0 =$ strong minimizer

PROOF

suppose there exist two strong global minimizers: x_1, x_2

$f(x_1) < f(x_2) \Rightarrow f(x_2) < f(x_1) \Rightarrow x_1 = x_2$ (the point is unique)

PRESERVATION OF MAXIMIZERS/MINIMIZERS WITH RESPECT TO STRICTLY INCREASING COMPOSITION

let $g: B \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a strictly increasing function with $\text{Im } f \subseteq B$
the two optimization problems

$\max_x f(x) \text{ sub } x \in C$ and $\max_x (g \circ f)(x) \text{ sub } x \in C$
are equivalent meaning they have the same solution

PROOF

since g strictly increasing by strict monotonicity being equivalent to an order isomorphism we have $s \geq t \Leftrightarrow g(s) \geq g(t) \quad \forall s, t \in \text{Im } f$

thus $\underbrace{f(x)}_s \geq \underbrace{f(y)}_t \Leftrightarrow g(f(x)) \geq g(f(y)) \quad \forall x, y \in A$

therefore $\forall x \in C, f(\hat{x}) \geq f(x) \Leftrightarrow \forall x \in C, (g \circ f)(\hat{x}) \geq (g \circ f)(x)$

this proves that a vector $\hat{x} \in C$ solves $\max_x f(x) \text{ sub } x \in C$ if and only if it solves $\max_x (g \circ f)(x) \text{ sub } x \in C$

FENCHEL

given $f: C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, A convex subset, f concave

if $x_0 \in A$ it is a local maximizer \Rightarrow global maximizer

PROOF

if $x_0 =$ local maximizer, then \forall a neighbourhood $B_\epsilon(x_0) \Rightarrow f(x_0) \geq f(x)$

$\forall x \in B_\epsilon(x_0) \Rightarrow x_0 =$ global maximizer

suppose $x_0 \neq$ global maximizer then $\forall y \in A$ such that $f(y) > f(x_0)$

since f concave $\forall t \in [0, 1] \Rightarrow$

$$f(tx_0 + (1-t)y) \geq tf(x_0) + (1-t)f(y) > tf(x_0) + (1-t)f(x_0) > f(x_0)$$

since A convex $\Rightarrow tx_0 + (1-t)y \in A \quad \forall t \in [0, 1]$

then $\exists t' \in \mathbb{R}$ such that: $tx_0 + (1-t)y \in B_\epsilon(x_0) \quad \forall t \in [t', 1] \Rightarrow$

$\forall t \in [t', 1], f(tx_0 + (1-t)y) > f(x_0) =$ there exist points of $B_\epsilon(x_0)$ where $f(x) > f(x_0)$

this contradicts $f(x_0) \geq f(x) \quad \forall x \in B_\epsilon(x_0) \Rightarrow$

$x_0 =$ global maximizer

CONCAVE FUNCTIONS HAVE CONVEX UPPER CONTOUR SETS

if $f: C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is concave then all its upper contour sets $(f \geq k)$ are convex

PROOF

given $k \in \mathbb{R}$ let $(f \geq k)$ be non-empty (otherwise the result is obvious because empty sets are trivially convex)

let $x, y \in (f \geq k)$ and $\alpha \in [0, 1]$

by the concavity of f : $f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y) \geq \alpha k + (1-\alpha)k = k$

therefore $\alpha x + (1-\alpha)y \in (f \geq k)$

CONVEX FUNCTIONS ARE QUASI-CONVEX

$$\forall x, y \in A$$

$$f[\alpha x + (1-\alpha)y] \leq \alpha f(x) + (1-\alpha)f(y) \text{ if } f(x) \geq f(y) \text{ then}$$

$$\alpha f(x) + (1-\alpha)f(y) \leq \alpha f(x) + (1-\alpha)f(x) \leq f(x)$$

\Rightarrow

$$\text{if } f(y) \geq f(x) \text{ then } \alpha f(x) + (1-\alpha)f(y) \leq f(y)$$

\Rightarrow

$$\alpha f(x) + (1-\alpha)f(y) \leq \max\{f(x), f(y)\} =$$

$$f[\alpha x + (1-\alpha)y] \leq \max\{f(x), f(y)\} = f \text{ quasi-convex}$$

CONCAVE FUNCTIONS ARE QUASI-CONCAVE

$$\forall x, y \in A$$

$$f[\alpha x + (1-\alpha)y] \geq \alpha f(x) + (1-\alpha)f(y) \text{ if } f(x) \leq f(y) \text{ then}$$

$$\alpha f(x) + (1-\alpha)f(y) \geq \alpha f(x) + (1-\alpha)f(x) \geq f(x)$$

\Rightarrow

$$\text{if } f(y) \geq f(x) \text{ then } \alpha f(x) + (1-\alpha)f(y) \geq f(x)$$

\Rightarrow

$$\alpha f(x) + (1-\alpha)f(y) \geq \min\{f(x), f(y)\} =$$

$$f[\alpha x + (1-\alpha)y] \geq \min\{f(x), f(y)\} = f \text{ quasi-concave}$$

sequences

- a law defined on natural numbers which produces an infinite list
- number sequence = function $f: \mathbb{N} \rightarrow \mathbb{R}$ (domain: \mathbb{N} + conditions, codomain: $\mathbb{R} \rightarrow$ irrelevant)
- $f, g \rightarrow a, b$ $f(x), g(x) \rightarrow a_n, b_n, x_n, y_n$
 $a_n = 0 (a_0), 1 (a_1), 4 (a_2), \dots$

RECURSIVE SEQUENCES

$$\begin{cases} a_0 \text{ given} \\ a_{n+1} = f(a_n) \end{cases} \quad \forall n \geq 0 \rightarrow \text{Fibonacci sequence} = \begin{cases} a_0 = 1 \\ a_1 = 1 \\ a_{n+2} = a_n + a_{n+1} \end{cases} \quad \forall n \geq 0$$

$\hookrightarrow a_n = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$

LIMITS OF SEQUENCES

- ① $\lim_{n \rightarrow +\infty} a_n = l \rightarrow a_n$ converges to l
 a_n tends to l
 - def. with neighborhoods $\rightarrow \forall$ neighborhood $B(l) \exists \bar{n} \in \mathbb{N} : \forall n \geq \bar{n} a_n \in B(l)$
 - def. with inequalities $\rightarrow \forall \varepsilon > 0, \exists \bar{n} \in \mathbb{N} : \forall n \geq \bar{n}, l - \varepsilon < a_n < l + \varepsilon$
- ② $\lim_{n \rightarrow +\infty} a_n = +\infty \rightarrow a_n$ diverges to $+\infty$
 a_n tends to $+\infty$
 - def. with neighborhoods $\rightarrow \forall$ neighborhood $B(+\infty) \exists \bar{n} \in \mathbb{N} : \forall n \geq \bar{n} a_n \in B(+\infty)$
 - def. with inequalities $\rightarrow \forall k > 0, \exists \bar{n} \in \mathbb{N} : \forall n \geq \bar{n}, a_n > k$
- ③ $\lim_{n \rightarrow +\infty} a_n = -\infty \rightarrow a_n$ diverges to $-\infty$
 a_n tends to $-\infty$
 - def. with neighborhoods $\rightarrow \forall$ neighborhood $B(-\infty) \exists \bar{n} \in \mathbb{N} : \forall n \geq \bar{n} a_n \in B(-\infty)$
 - def. with inequalities $\rightarrow \forall k > 0, \exists \bar{n} \in \mathbb{N} : \forall n \geq \bar{n}, a_n < -k$
- ④ $\lim_{n \rightarrow +\infty} a_n = \#$ = irregular / oscillating \rightarrow if a_n is not convergent, divergent to $+\infty$ or $-\infty$

LIMITS FROM BELOW / ABOVE

① $\lim_{n \rightarrow +\infty} a_n = l^- \rightarrow$ from below

- def. with neighborhoods $\rightarrow \forall$ left neighborhood $B^-(l) \exists \bar{n} \in \mathbb{N} : \forall n \geq \bar{n} a_n \in B^-(l)$ $r \rightarrow (l-\varepsilon, l]$
- def. with inequalities $\rightarrow \forall \varepsilon > 0, \exists \bar{n} \in \mathbb{N} : \forall n \geq \bar{n}, l-\varepsilon < a_n \leq l$

② $\lim_{n \rightarrow +\infty} a_n = l^+ \rightarrow$ from above

- def. with neighborhoods $\rightarrow \forall$ right neighborhood $B^+(l) \exists \bar{n} \in \mathbb{N} : \forall n \geq \bar{n}, a_n \in B^+(l)$ $r \rightarrow [l, l+\varepsilon)$
- def. with inequalities $\rightarrow \forall \varepsilon > 0, \exists \bar{n} \in \mathbb{N} : \forall n \geq \bar{n}, l \leq a_n < l+\varepsilon$

L

- a sequence is a function $f: \mathbb{N} \rightarrow \mathbb{R}$, so it is a particular case of $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$

so \rightarrow • constant sequences

• bounded sequences

• increasing / decreasing / strictly increasing / strictly decreasing sequences \rightarrow ex. increasing = $\forall m, n \in \mathbb{N}, m > n, a_m \geq a_n$

$\rightarrow \forall n \in \mathbb{N}, a_{n+1} \geq a_n$

ELEMENTARY SEQUENCES

① $a_n = n^\alpha \rightarrow$ power sequences

- $\alpha > 0 = \rightarrow +\infty$ ($n^2, n^{\frac{1}{2}} = \sqrt{n}, \dots$)

- $\alpha = 0 = \rightarrow 1$ ($1, 1, 1, 1, \dots$)

- $\alpha < 0 = \rightarrow 0^+$ ($\frac{1}{n}, n^{-q}, \dots$)

② $a_n = q^n \rightarrow$ exponential sequences

- $q > 1 = \rightarrow +\infty$ ($2^n, 10^n, e^n, \dots$)

- $q = 1 = \rightarrow 1$ ($1, 1, 1, 1, \dots$)

- $-1 < q < 1 = \rightarrow 0$

• irr. bounded = $q = \pm 1 \rightarrow a_n = 1, -1,$

$1, -1, \dots$

• irr. unbounded = $q < -1 \rightarrow (-2)^n,$

$(-10)^n, \dots$

$$\left\{ \begin{array}{l} 0 < q < 1 = 0^+ \left(\left(\frac{1}{2}\right)^n, \left(\frac{1}{10}\right)^n, \dots \right) \\ q = 0 = 0 \left(0, 0, 0, 0, \dots \right) \\ -1 < q < 1 = 0 \left(\left(-\frac{1}{2}\right)^n, \dots \right) \end{array} \right.$$

③ $a_n = (\ln n)^\alpha \rightarrow$ logarithmic sequences

- $\alpha > 0 \rightarrow +\infty ((\ln n)^2, (\ln n)^{\frac{1}{2}}, \dots)$

- $\alpha = 0 \rightarrow 1 (1, 1, 1, 1, \dots)$

- $\alpha < 0 \rightarrow 0^+ ((\ln n)^{-2}, \dots)$

GEOMETRIC SEQUENCES

- $a_n = aq^n \rightarrow a = 5, q = 2 \rightarrow a_n = 5, 5 \cdot 2, 5 \cdot 2^2, 5 \cdot 2^3, \dots$

- recursive $\rightarrow \begin{cases} a_0 = a \\ \frac{a_{n+1}}{a_n} = q \end{cases} \rightarrow \begin{cases} a_0 = a \\ a_{n+1} = qa_n \end{cases} \rightarrow a = 5, q = 2$
 $a_n = 5, 5 \cdot 2, 5 \cdot 2^2, 5 \cdot 2^3, \dots$
constant ratio

ARITHMETIC SEQUENCES

- $a_n = a + dn$ (linear sequences) $\rightarrow a = 10, d = 4 \rightarrow a_n = 10, 14, 18, 22, \dots$

- recursive $\rightarrow \begin{cases} a_0 = a \\ a_{n+1} - a_n = d \end{cases} \rightarrow \begin{cases} a_0 = a \\ a_{n+1} = a_n + d \end{cases} \rightarrow a = 10, d = 4$
 $a_n = 10, 14, 18, 22, \dots$
constant difference

OPERATIONS WITH LIMITS

consider x_n, y_n such that $\exists \lim_{n \rightarrow +\infty} x_n = L, \lim_{n \rightarrow +\infty} y_n = \pi$, with $L, \pi \in \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ then:

① $\lim_{n \rightarrow +\infty} (x_n + y_n) = \lim_{n \rightarrow +\infty} x_n + \lim_{n \rightarrow +\infty} y_n$

② $\lim_{n \rightarrow +\infty} (x_n \cdot y_n) = \lim_{n \rightarrow +\infty} x_n \cdot \lim_{n \rightarrow +\infty} y_n$

③ $\lim_{n \rightarrow +\infty} \frac{x_n}{y_n} = \frac{\lim_{n \rightarrow +\infty} x_n}{\lim_{n \rightarrow +\infty} y_n}$

provided that there is no indeterminate form:

- $+\infty, -\infty$

- $0 \cdot (+\infty), 0 \cdot (-\infty) \rightarrow 0 \cdot \infty$

- $\frac{0}{0}$

- $\frac{+\infty}{+\infty}, \frac{+\infty}{-\infty}, \frac{-\infty}{+\infty}, \frac{-\infty}{-\infty} \rightarrow \frac{\infty}{\infty}$

if we do not get an indeterminate form the limit is easy; if we get an indeterminate form, we have to analyse it better

When one of the sequences x_n, y_n is irregular the theorems cannot be applied but there are other methods \rightarrow

$$\lim_{n \rightarrow +\infty} (n^2 + (-1)^n) = +\infty$$

$\underbrace{\quad}_{\rightarrow +\infty} \quad \underbrace{\quad}_{\text{irregular bounded}}$

there is also a specific theorem

COMPARISON CRITERION (aka "the two policemen" / "squeeze theorem")

suppose $x_n \leq y_n \leq z_n$ (at least eventually)

$$\text{suppose } \lim_{n \rightarrow +\infty} x_n = L, \lim_{n \rightarrow +\infty} z_n = L \rightarrow \lim_{n \rightarrow +\infty} y_n = L$$

RATIO CRITERION (only the following limit form)

consider a sequence x_n

$$\text{if } \exists q \in \mathbb{R}, q < 1, \text{ such that: } \lim_{n \rightarrow +\infty} \left| \frac{x_{n+1}}{x_n} \right| = q \text{ then } x_n \rightarrow 0$$

INDETERMINATE FORMS

suppose $a_n, b_n \rightarrow +\infty$ (if they go to $-\infty$ it is similar) then:

- ① if $\lim_{n \rightarrow +\infty} \frac{a_n}{b_n} = +\infty$, we say that: a_n is an infinity of higher order with respect to b_n = "faster"
- ② if $\lim_{n \rightarrow +\infty} \frac{a_n}{b_n} = 0^+$, we say that: a_n is an infinity of lower order with respect to b_n = "slower"
- ③ if $\lim_{n \rightarrow +\infty} \frac{a_n}{b_n} = k \neq 0$, we say that: a_n, b_n are infinities of the same order = same speed
- ④ if $\lim_{n \rightarrow +\infty} \frac{a_n}{b_n} \nexists$, we say that: a_n, b_n are not comparable

THEOREMS

suppose $p^n, q^n, n^\alpha, n^\beta, (\ln n)^\alpha, (\ln n)^\beta$ are such that they $\rightarrow +\infty$ [$p, q > 1$; $\alpha, \beta > 0$] then:

- ① $\lim_{n \rightarrow +\infty} \frac{n^\alpha}{n^\beta} = +\infty \leftrightarrow \alpha > \beta$ = comparison of powers
- ② $\lim_{n \rightarrow +\infty} \frac{p^n}{q^n} = +\infty \leftrightarrow p > q$ = comparison of exponentials
- ③ $\lim_{n \rightarrow +\infty} \frac{(\ln n)^\alpha}{(\ln n)^\beta} = +\infty \leftrightarrow \alpha > \beta$ = comparison of logarithms
- ④ $\lim_{n \rightarrow +\infty} \frac{q^n}{n^\alpha} = +\infty \forall q, \alpha$ = comparison exponential/power

THEOREM

suppose $a_n, b_n \rightarrow +\infty$

then $\frac{1}{a_n}, \frac{1}{b_n} \rightarrow 0^+$ and:

a_n is an infinity higher order with respect to b_n ($\rightarrow +\infty$ faster)
 \leftrightarrow

$\frac{1}{a_n}$ is an infinitesimal of higher order with respect to $\frac{1}{b_n}$ ($\rightarrow 0^+$ faster)

THE SEQUENCE $(1 + \frac{1}{n})^n$

take $a_n = (1 + \frac{1}{n})^n \rightarrow$ it converges to a finite limit L , $2 \leq L \leq 3$ which is

exactly $L = \sup \{a_n\} \downarrow$
 $e = \lim_{n \rightarrow +\infty} (1 + \frac{1}{n})^n$ (Napier's number) = $e \approx 2.71828$

generalisation $\rightarrow \lim_{n \rightarrow +\infty} (1 + \frac{a}{n})^n = e^a \quad \forall a \in \mathbb{R}$

LAUNDAU SYMBOLS: $\sim, \sim\sim, \sim\sim\sim$

ASYMPTOTIC \sim

a_n, b_n sequences $a_n \sim b_n$ if: $\lim_{n \rightarrow +\infty} \frac{a_n}{b_n} = 1 \rightarrow a_n, b_n$ tend to the same limit with exactly the same speed \downarrow

- you can reduce complicated sequences to a simpler sequence with the same behavior

it is only an intuitive interpretation (formally valid only if $\lim a_n, \lim b_n \exists$)

- \sim is useful to compute limits more easily

LITTLE O

a_n, b_n sequences $a_n = o(b_n)$ if: $\lim_{n \rightarrow +\infty} \frac{a_n}{b_n} = 0 \rightarrow a_n$ is negligible with respect to b_n

THEOREM

$x_n \sim y_n \leftrightarrow x_n = y_n + o(y_n)$ = relationship between \sim and o

SEQUENCES OF VECTORS

a sequence of vectors x_n with m components is a function $f: \mathbb{N} \rightarrow \mathbb{R}^m \rightarrow$
 $\lim_{n \rightarrow +\infty} x_n = \underline{l} \rightarrow \forall$ neighborhood $B(\underline{l}) \exists \bar{n} \in \mathbb{N}: \forall n \geq \bar{n}, x_n \in B(\underline{l})$

consider a sequence of vectors with m components $x_n = (x_{1n}, x_{2n}, \dots, x_{mn})$
we have that $x_n \rightarrow \underline{l} = (l_1, l_2, \dots, l_m) \leftrightarrow \begin{cases} x_{1n} \rightarrow l_1 \\ x_{2n} \rightarrow l_2 \\ \vdots \\ x_{mn} \rightarrow l_m \end{cases}$

theorems and proofs for sequences

FOR AN EVENTUALLY POSITIVE SEQUENCE $\{x_n\}$ DIVERGES TO $+\infty \Leftrightarrow \{\frac{1}{x_n}\}$ CONVERGES TO 0^+

PROOF

if $x_n \rightarrow +\infty$ then $\forall \varkappa > 0 \exists n_0 : \forall n \geq n_0, x_n > \varkappa$

$$\varepsilon = \frac{1}{\varkappa}, x_n > \varkappa \rightarrow x_n > \frac{1}{\varepsilon} \Rightarrow 0 < \frac{1}{x_n} < \varepsilon$$

$$\Rightarrow \forall \varepsilon > 0 \exists n_0 : \forall n \geq n_0, 0 < \frac{1}{x_n} < \varepsilon = \frac{1}{x_n} \rightarrow 0^+$$

$$\frac{1}{x_n} \rightarrow 0^+ = \forall \varepsilon > 0 \exists n_0 : \forall n \geq n_0, 0 < \frac{1}{x_n} < \varepsilon$$

$$\varepsilon = \frac{1}{\varkappa} \rightarrow 0 < \frac{1}{x_n} < \varepsilon \Rightarrow 0 < \frac{1}{x_n} < \frac{1}{\varkappa} \Rightarrow x_n > \varkappa$$

$$\forall \varkappa > 0 \exists n_0 : \forall n \geq n_0, x_n > \varkappa \Rightarrow x_n \rightarrow +\infty$$

THEOREM ON THE UNIQUENESS OF THE LIMIT (FINITE LIMITS)

a sequence $\{x_n\}$ converges at most to one limit $l \in L$

PROOF

suppose $\lim_{n \rightarrow +\infty} a_n = l_1$, $\lim_{n \rightarrow +\infty} a_n = l_2$, $l_1 \neq l_2$

if $\varepsilon < \frac{l_2 - l_1}{2}$ then $B_\varepsilon(l_1) \cap B_\varepsilon(l_2) = \emptyset$

if $\lim_{n \rightarrow +\infty} a_n = l_1$ then $\exists n'_0$ such that $\forall n \geq n'_0, x_n \in B_\varepsilon(l_1)$

if $\lim_{n \rightarrow +\infty} a_n = l_2$ then $\exists n''_0$ such that $\forall n \geq n''_0, x_n \in B_\varepsilon(l_2)$

$\Rightarrow \forall n > \max$ between n'_0 and $n''_0, x_n \in B_\varepsilon(l_1) \cap B_\varepsilon(l_2)$

↓
contradiction to $B_\varepsilon(l_1) \cap B_\varepsilon(l_2) = \emptyset \Rightarrow l_1 = l_2$ unique

BOUNDLESS THEOREM FOR CONVERGENT SEQUENCES

each convergent sequence is bounded

PROOF

suppose $x_n \rightarrow L$, setting $\epsilon = 1$ there exists $n_1 > 1$ such that $x_n \in B_1(L)$ for every $n \geq n_1$,

let $\pi > 0$ be a constant such that $\pi > \max [1, d(x_1, L), \dots, d(x_{n_1-1}, L)]$

we have $d(x_n, L) < \pi$ for every $n \geq 1$, i.e. $|x_n - L| < \pi$ for every $n \geq 1$

this implies that for all $n \geq 1$, $L - \pi < x_n < L + \pi$

therefore the sequence is bounded

REGULARITY THEOREM FOR MONOTONE SEQUENCES

each monotone sequence is regular, in particular:

- ① it converges if it is bounded
- ② it diverges positively if it is increasing and unbounded
- ③ it diverges negatively if it is decreasing and unbounded

PROOF

let $\{x_n\}$ be an increasing sequence

it can be either bounded or unbounded above (for sure it is bounded below because $x_1 \leq x_n$ for every $n \geq 1$)

suppose that $\{x_n\}$ is bounded, we want to prove it is convergent

let E be the image of the sequence

by hypothesis it is a bounded subset of \mathbb{R}

by the Least Upper Bound Principle, $\sup E$ exists

set $L = \sup E$, we prove $x_n \rightarrow L$, let $\varepsilon > 0$

since L is the supremum of E by characterization of the supremum we have:

① $L \geq x_n$ for every $n \geq 1$, ② there exists an element x_{n_ε} of E such that

$$x_{n_\varepsilon} > L - \varepsilon$$

since $\{x_n\}$ is an increasing sequence it then follows that $L \geq x_n \geq x_{n_\varepsilon} > L - \varepsilon$

$$\forall n \geq n_\varepsilon$$

hence $x_n \in B_\varepsilon(L)$ for every $n \geq n_\varepsilon$ as desired

suppose that $\{x_n\}$ is unbounded above

then for every $k > 0$ there exists an element x_{n_k} such that $x_{n_k} > k$

since $\{x_n\}$ is increasing we have that $x_n \geq x_{n_k} > k$ for every $n \geq n_k$ so it diverges to $+\infty$

LIMIT OF THE SUM OF SEQUENCES

let $x_n \rightarrow L \in \bar{\mathbb{R}}$ and $y_n \rightarrow H \in \bar{\mathbb{R}}$ then:

$x_n + y_n \rightarrow L + H$ provided that $L + H$ is not an indeterminate form of the type $+\infty - \infty$ or $-\infty + \infty$

PROOF

let $x_n \rightarrow L$ and $y_n \rightarrow H$ with $L, H \in \mathbb{R}$

this means that for every $\epsilon > 0$ there exist n_1 and n_2 such that:

$$L - \epsilon < x_n < L + \epsilon \quad \forall n \geq n_1 \quad \text{and} \quad H - \epsilon < y_n < H + \epsilon \quad \forall n \geq n_2$$

by adding inequalities member by member for every $n \geq n_3 = \max\{n_1, n_2\}$

$$\text{we have } L + H - 2\epsilon < x_n + y_n < L + H + 2\epsilon$$

since 2ϵ is arbitrary it follows that $x_n + y_n \rightarrow L + H$

COMPARISON CRITERION FOR SEQUENCES

$\{x_n\}, \{y_n\}, \{z_n\}$

$$y_n \leq x_n \leq z_n \Rightarrow \lim_{n \rightarrow +\infty} y_n = \lim_{n \rightarrow +\infty} z_n = l \in \mathbb{R} \Rightarrow \lim_{n \rightarrow +\infty} x_n = l$$

PROOF

$$y_n \rightarrow l \Rightarrow \forall \varepsilon > 0 \exists n_1 : l - \varepsilon < y_n < l + \varepsilon \quad \forall n \geq n_1,$$

$$z_n \rightarrow l \Rightarrow \forall \varepsilon > 0 \exists n_2 : l - \varepsilon < z_n < l + \varepsilon \quad \forall n \geq n_2$$

$$\forall n \geq n_3 \Rightarrow y_n \leq x_n \leq z_n \Rightarrow \forall n > \max\{n_1, n_2, n_3\} :$$

$$l - \varepsilon < y_n \leq x_n \leq z_n < l + \varepsilon \Rightarrow l - \varepsilon < x_n < l + \varepsilon \Rightarrow x_n \rightarrow l$$

THEOREM ON THE CHARACTERIZATION OF $x_n \sim y_n$ WITH THE USE OF THE LITTLE- o

we have $x_n \sim y_n \Leftrightarrow x_n = y_n + o(y_n)$

meaning two sequences are asymptotic when they are equal up to a component that is asymptotically negligible with respect to them

this result further clarifies how the relation \sim can be seen as an asymptotic equality

we can also write $x_n \sim y_n \Leftrightarrow x_n = y_n [1 + o(1)]$

PROOF

if \Rightarrow from $x_n = y_n + o(y_n)$ it follows that $\frac{x_n}{y_n} = \frac{y_n + o(y_n)}{y_n} = 1 + \frac{o(y_n)}{y_n} \rightarrow 1$

only if \Rightarrow let $x_n \sim y_n$

denoting $z_n = x_n - y_n$ one has that $\frac{z_n}{y_n} = \frac{x_n - y_n}{y_n} = \frac{x_n}{y_n} - 1 \rightarrow 0$
therefore $z_n = o(y_n)$

series

- a series $\sum_{n=0}^{+\infty} a_n$ is the sum of the infinitely many terms of a sequence a_n
- DEF. consider a sequence $a_n = a_0, a_1, a_2, a_3, \dots$
 - ① we define a second sequence S_n , called the sequence of partial sums of a_n :
$$S_0 = a_0$$
$$S_1 = a_0 + a_1$$
$$S_2 = a_0 + a_1 + a_2$$
$$\vdots$$
$$S_n = a_0 + a_1 + a_2 + \dots + a_n \rightarrow \forall n, \text{ it is a finite sum}$$
 - ② we call series with general term a_n and write $\sum_{n=0}^{+\infty} a_n$ the limit of the sequence S_n when $n \rightarrow +\infty$ if this limit exists:

$$\sum_{n=0}^{+\infty} a_n = \lim_{n \rightarrow +\infty} S_n$$

- four possible behaviors \rightarrow
 - if $\lim_{n \rightarrow +\infty} S_n = s \rightarrow \sum_{n=0}^{+\infty} a_n = s$ (converges to s / has sum s)
 - if $\lim_{n \rightarrow +\infty} S_n = +\infty \rightarrow \sum_{n=0}^{+\infty} a_n = +\infty$ (diverges to $+\infty$)
 - if $\lim_{n \rightarrow +\infty} S_n = -\infty \rightarrow \sum_{n=0}^{+\infty} a_n = -\infty$ (diverges to $-\infty$)
 - if the sequence S_n is irregular $\rightarrow \sum_{n=0}^{+\infty} a_n$ is irregular

GEOMETRIC SERIES

$$a_n = aq^n \text{ (geometric sequence)} \rightarrow \sum_{n=0}^{+\infty} (aq^n) \text{ (geometric series)}$$

THEOREM = behavior of the geometric series

$$\sum_{n=0}^{+\infty} q^n \begin{cases} \text{converges to } \frac{1}{1-q} & \text{if } -1 < q < 1 \\ \text{diverges to } +\infty & \text{if } q \geq 1 \\ \text{is irregular} & \text{if } q \leq -1 \end{cases}$$

THEOREM

suppose $\sum_{n=0}^{+\infty} a_n$ is regular and $k \in \mathbb{R}$ then also $\sum_{n=0}^{+\infty} k a_n$ is regular and in the convergent case it is $\sum_{n=0}^{+\infty} k a_n = k \sum_{n=0}^{+\infty} a_n$

KEUNGOLI SERIES (THEOREM)

$$\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = 2$$

HARMONIC SERIES (THEOREM)

given the sequence $a_n = \frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

the series $\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ (harmonic series) is divergent to $+\infty$:
 $\sum_{n=1}^{+\infty} \frac{1}{n} = +$

NECESSARY CONDITION FOR CONVERGENCE (THEOREM)

if $\sum_{n=0}^{+\infty} a_n$ is convergent to s then $a_n \rightarrow 0$ or the fact that $a_n \rightarrow 0$ is a necessary condition for the convergence of the series $\sum_{n=0}^{+\infty} a_n$

equivalent statement = if $a_n \not\rightarrow 0$ then $\sum_{n=0}^{+\infty} a_n$ is not convergent (maybe divergent to $+\infty$ / $-\infty$ or irregular)

REGULARITY THEOREM

- consider $\sum_{n=0}^{+\infty} a_n$ with $a_n \geq 0$ at least $\forall n \geq \bar{n}$ then the series is regular \rightarrow either it converges to s or it diverges to $+\infty$
- if $a_n \geq 0$ and $a_n \not\rightarrow 0 \rightarrow$ the series diverges to $+\infty$

BEHAVIOUR OF THE GENERALIZED HARMONIC SERIES

consider the generalized harmonic series $\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$, $\alpha \in \mathbb{R}$ then:

$\left\{ \begin{array}{l} \text{it converges} \quad \forall \alpha > 1 \\ \text{it diverges to } +\infty \quad \forall \alpha \leq 1 \end{array} \right.$

- the harmonic series diverges to $+\infty$
- among the series of this family \rightarrow
 - those for which $a_n \rightarrow 0$ quicker it converges (do not know to which number)
 - those for which $a_n \rightarrow 0$ slower it diverges to $+\infty$
- consider the series $\sum_{n=2}^{+\infty} \frac{1}{n^\alpha (\ln n)^\beta}$, $\alpha, \beta \in \mathbb{R}$ (which for $\beta = 0$ reduces to the generalized harmonic series) then:

$\left\{ \begin{array}{l} \text{it converges} \quad \forall \alpha > 1, \forall \beta \text{ and for} \\ \quad \alpha = 1, \beta > 1 \\ \text{it diverges to } +\infty \quad \forall \alpha < 1, \forall \beta \text{ and for} \\ \quad \alpha = 1, \beta \leq 1 \end{array} \right.$

THRESHOLD SERIES

$\sum_{n=2}^{+\infty} \frac{1}{n \ln n}$ which diverges to $+\infty$

among the series of this family \rightarrow

- those for which $a_n \rightarrow 0$ quicker it converges
- those for which $a_n \rightarrow 0$ slower it diverges to $+\infty$

ASYMPTOTIC COMPARISON CRITERION (THEOREM)

- consider $\sum_{n=0}^{+\infty} a_n, \sum_{n=0}^{+\infty} b_n$ with $a_n, b_n \geq 0$ (at least eventually)
- if $a_n \sim b_n$ then:
either $\sum_{n=0}^{+\infty} a_n, \sum_{n=0}^{+\infty} b_n$ are both convergent (not necessarily to the same number)
or they are both divergent to $+\infty$
- it is the most useful criterion but it cannot always be applied

COMPARISON CRITERION (THEOREM)

- consider $\sum_{n=0}^{+\infty} a_n, \sum_{n=0}^{+\infty} b_n$ with $a_n, b_n \geq 0$ (at least eventually)
- suppose $a_n \leq b_n$ (at least eventually) then:
 - if $\sum_{n=0}^{+\infty} b_n$ converges $\rightarrow \sum_{n=0}^{+\infty} a_n$ converges
 - if $\sum_{n=0}^{+\infty} a_n$ diverges to $+\infty \rightarrow \sum_{n=0}^{+\infty} b_n$ diverges to $+\infty$

OTHER SERIES

$\sum_{n=0}^{+\infty} a_n$ with $a_n \leq 0$ (at least $\forall n \geq \bar{n}$) \rightarrow ex. $\sum_{n=1}^{+\infty} \left(-\frac{1}{n^2}\right) = -\sum_{n=1}^{+\infty} \frac{1}{n^2}$ (it converges)

you collect the minus and study using all the preceding case

SERIES WITH TERMS OF INDEFINITE SIGN

given $\sum_{n=0}^{+\infty} a_n$ with a_n of indefinite sign we say that $\sum_{n=0}^{+\infty} a_n$ is absolutely convergent if the series $\sum_{n=0}^{+\infty} |a_n|$ is convergent

THEOREM

given $\sum_{n=0}^{+\infty} a_n$ with terms of indefinite sign if $\sum_{n=0}^{+\infty} |a_n|$ is absolutely convergent (that is: $\sum_{n=0}^{+\infty} |a_n|$ is convergent) then $\sum_{n=0}^{+\infty} a_n$ is also convergent (simply convergent)

REMARK ON EXPONENTIAL SERIES

$$\sum_{n=0}^{+\infty} \frac{1}{n!} = e \left(\frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots \right) \quad \Bigg| \quad \sum_{n=0}^{+\infty} \frac{a^n}{n!} = e^a \left(\frac{1}{1} + \frac{a}{1} + \frac{a^2}{2} + \frac{a^3}{6} + \dots \right)$$

theorems and proofs for series

BEHAVIOUR OF THE GEOMETRIC SERIES

$$\sum_{n=0}^{+\infty} q^n \begin{cases} \text{converges to } \frac{1}{1-q} & \text{if } -1 < q < 1 \\ \text{diverges to } +\infty & \text{if } q \geq 1 \\ \text{is irregular} & \text{if } q \leq -1 \end{cases}$$

PROOF

if $q=1$ then $S_n = 1 + q + q^2 + \dots + q^n = 1 + 1 + 1 + \dots + 1 = n+1 \rightarrow +\infty$
so $\sum q^n$ diverges to $+\infty$

if $S_n = 1 + q + q^2 + \dots + q^n$ then $q \cdot S_n = q + q^2 + q^3 + \dots + q^{n+1}$ and $S_n - qS_n = (1 + q + q^2 + \dots + q^n) - (q + q^2 + q^3 + \dots + q^{n+1}) = 1 - q^{n+1}$
so $S_n - q \cdot S_n = S_n \cdot (1 - q) = 1 - q^{n+1} \Rightarrow$ if $q \neq 1$ then $S_n = \frac{1 - q^{n+1}}{1 - q} \Rightarrow$

if $q > 1$ $q^{n+1} \rightarrow +\infty$ so $S_n \rightarrow +\infty$

if $|q| < 1$ $q^{n+1} \rightarrow 0$ so $S_n \rightarrow \frac{1}{1-q}$

if $q = -1 \Rightarrow$ num. 2 or 0, den. 0 $\Rightarrow S_n = \begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases} \Rightarrow$

$\nexists \lim_{n \rightarrow +\infty} S_n =$ the series is irregular

if $q < -1 \Rightarrow$ num. negative and positive term always growing (in absolute value), den. negative $\Rightarrow \nexists \lim_{n \rightarrow +\infty} S_n =$ the series is irregular

BEHAVIOUR OF THE MENGOLI SERIES

$$\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = 1 \Rightarrow \text{it converges and has sum } 1$$

PROOF

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \Rightarrow S_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1} = 1$$

BEHAVIOUR OF THE HARMONIC SERIES

$$\sum_{n=1}^{+\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \Rightarrow \text{diverges to } +\infty$$

PROOF

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \dots$$

$\underbrace{\hspace{10em}}_{\geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}} \quad \underbrace{\hspace{10em}}_{\geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}} \quad \underbrace{\hspace{10em}}_{> 8 \cdot \frac{1}{16} = \frac{1}{2}}$

$$S_2 = S_{2^1} = 1 + \frac{1}{2} \geq 1 + 1 \cdot \frac{1}{2}$$

$$\Rightarrow S_{2^k} > 1 + k \cdot \frac{1}{2}$$

$$S_4 = S_{2^2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2 \cdot \frac{1}{2}$$

$$\lim_{n \rightarrow +\infty} S_n = +\infty \rightarrow \sum_{n=1}^{+\infty} \frac{1}{n^k} \text{ diverges to } +\infty$$

NECESSARY CONDITION FOR CONVERGENCE

if the series $\sum_{n=1}^{\infty} x_n$ converges then $x_n \rightarrow 0$

PROOF

clearly we have $x_n = s_n - s_{n-1}$ and given that the series converges $s_n \rightarrow S$ as well

as $s_{n-1} \rightarrow S$

therefore $x_n = s_n - s_{n-1} \rightarrow S - S = 0$

REGULARITY THEOREM

each series with positive terms is either convergent or positively divergent
it cannot be indeterminate

PROOF

consider $\sum_{n=1}^{\infty} x_n$ where $x_n \geq 0 \quad \forall n$

then $S_n = \sum_{i=1}^n$ is increasing then monotone

a monotonic sequence (one either increasing or decreasing) on \mathbb{R} must either converge to a finite limit or diverge to $+\infty$ if the sequence is increasing

thus the series is either convergent or positively divergent

we conclude saying the series cannot be indeterminate

limits

take $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, we define \rightarrow

① given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, suppose c is a limit point of A (no need that $c \in A$) then

$\lim_{x \rightarrow c} f(x) = l$ means:

- \forall neighborhood $\mathcal{U}(l) \exists$ a neighborhood $\mathcal{V}(c)$ such that if $x \in \mathcal{V}(c)$, $x \neq c$ then $f(x) \in \mathcal{U}(l)$ = using neighborhoods
- $\forall \epsilon > 0 \exists \delta = \delta(\epsilon) > 0$ such that if $c - \delta < x < c + \delta$, $x \neq c$ then $l - \epsilon < f(x) < l + \epsilon$ = using inequalities

② given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, suppose c is a limit point of A (no need that $c \in A$) then

$\lim_{x \rightarrow c} f(x) = +\infty$ means:

- \forall neighborhood $\mathcal{U}(+\infty) \exists$ a neighborhood $\mathcal{V}(c)$ such that if $x \in \mathcal{V}(c)$, $x \neq c$ then $f(x) \in \mathcal{U}(+\infty)$ = using neighborhoods
- $\forall M > 0 \exists \delta = \delta(M) > 0$ such that if $c - \delta < x < c + \delta$, $x \neq c$ then $f(x) > M$ = using inequalities

③ given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, suppose c is a limit point of A (no need that $c \in A$) then

$\lim_{x \rightarrow c} f(x) = -\infty$ means:

- \forall neighborhood $\mathcal{U}(-\infty) \exists$ a neighborhood $\mathcal{V}(c)$ such that if $x \in \mathcal{V}(c)$, $x \neq c$ then $f(x) \in \mathcal{U}(-\infty)$ = using neighborhoods
- $\forall N < 0 \exists \delta = \delta(N) > 0$ such that if $c - \delta < x < c + \delta$, $x \neq c$ then $f(x) < N$ = using inequalities

$\lim_{x \rightarrow c^+} f(x) = l, +\infty, -\infty$ ($l \in \bar{\mathbb{R}}$) on the right (\uparrow -side limit) \rightarrow

• $-\infty$ = given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ suppose c is a limit point of A $\lim_{x \rightarrow c^+} f(x) = -\infty$ means:

- \forall neighborhood $\mathcal{U}(-\infty) \exists$ a right neighborhood $\mathcal{V}^+(c)$ such that if $x \in \mathcal{V}^+(c)$, $x \neq c$ then $f(x) \in \mathcal{U}(-\infty)$
- $\forall N < 0 \exists \delta = \delta(N) > 0$ such that if $c < x < c + \delta$ then $f(x) < N$

• $+\infty$ = given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ suppose c is a limit point of A $\lim_{x \rightarrow c^+} f(x) = +\infty$ means:

- \forall neighborhood $\mathcal{U}(+\infty) \exists$ a right neighborhood $\mathcal{V}^+(c)$ such that if $x \in \mathcal{V}^+(c)$, $x \neq c$ then $f(x) \in \mathcal{U}(+\infty)$
- $\forall M > 0 \exists \delta = \delta(M) > 0$ such that if $c < x < c + \delta$ then $f(x) > M$

• l = given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ suppose c is a limit point of A $\lim_{x \rightarrow c^+} f(x) = l$ means:

- \forall neighborhood $\mathcal{U}(l) \exists$ a right neighborhood $\mathcal{V}^+(c)$ such that if $x \in \mathcal{V}^+(c)$, $x \neq c$ then $f(x) \in \mathcal{U}(l)$

- $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0$ such that if $c - \delta < x < c + \delta$, $x \neq c$ then $l - \varepsilon < f(x) < l + \varepsilon$

- $\lim_{x \rightarrow c^-} f(x) = l, +\infty, -\infty$ ($l \in \mathbb{R}$) on the left (1-side limit) \rightarrow

• $-\infty$ = given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ suppose c is a limit point of A $\lim_{x \rightarrow c^-} f(x) = -\infty$
means:

- \forall neighborhood $\mathcal{U}(-\infty) \exists$ a left neighborhood $V^-(c)$ such that if $x \in V^-(c)$, $x \neq c$ then $f(x) \in \mathcal{U}(-\infty)$

- $\forall \pi > 0 \exists \delta = \delta(\pi) > 0$ such that $c - \delta < x < c$ then $f(x) < -\pi$

• $+\infty$ = given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ suppose c is a limit point of A $\lim_{x \rightarrow c^-} f(x) = +\infty$
means:

- \forall neighborhood $\mathcal{U}(+\infty) \exists$ a left neighborhood $V^-(c)$ such that if $x \in V^-(c)$, $x \neq c$ then $f(x) \in \mathcal{U}(+\infty)$

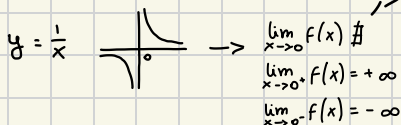
- $\forall \pi > 0 \exists \delta = \delta(\pi) > 0$ such that $c - \delta < x < c$ then $f(x) > \pi$

• l = given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ suppose c is a limit point of A $\lim_{x \rightarrow c^-} f(x) = l$
means:

- \forall neighborhood $\mathcal{U}(l) \exists$ a left neighborhood $V^-(c)$ such that if $x \in V^-(c)$, $x \neq c$ then $f(x) \in \mathcal{U}(l)$

- $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0$ such that if $c - \delta < x < c$, $x \neq c$ then $l - \varepsilon > f(x) > l + \varepsilon$

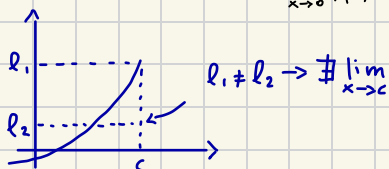
- when $\lim_{x \rightarrow c} f(x) \nexists$ but $\lim_{x \rightarrow c^-} f(x) \exists$ and $\lim_{x \rightarrow c^+} f(x) \exists$ it is still a nice function



THEOREMS

① $\exists \lim_{x \rightarrow c} f(x) = l \iff$

- $\exists \lim_{x \rightarrow c^-} f(x) = l_1$
- $\exists \lim_{x \rightarrow c^+} f(x) = l_2$
- $l_1 = l_2 = l$



② $\exists \lim_{x \rightarrow c} f(x) = +\infty \iff$

- $\exists \lim_{x \rightarrow c^-} f(x) = +\infty$
- $\exists \lim_{x \rightarrow c^+} f(x) = +\infty$

③ $\exists \lim_{x \rightarrow c} f(x) = -\infty \iff$

- $\exists \lim_{x \rightarrow c^-} f(x) = -\infty$
- $\exists \lim_{x \rightarrow c^+} f(x) = -\infty$

VERTICAL ASYMPTOTES

The straight line $x=c$ is a vertical asymptote for $y=f(x)$ if at least one of the following holds \rightarrow

- $\lim_{x \rightarrow c^-} f(x) = +\infty / -\infty$
- $\lim_{x \rightarrow c^+} f(x) = +\infty / -\infty$

$\lim_{x \rightarrow +\infty} f(x)$ AND $\lim_{x \rightarrow -\infty} f(x)$

- $\lim_{x \rightarrow +\infty} f(x) = l, +\infty, -\infty (l \in \bar{\mathbb{R}}) \rightarrow$

- $l =$ given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with A unbounded above $\lim_{x \rightarrow +\infty} f(x) = l$ means:
 - \forall neighborhood $\mathcal{U}(l) \exists$ a neighborhood $\mathcal{V}(+\infty)$ such that if $x \in \mathcal{V}(+\infty)$ then $f(x) \in \mathcal{U}(l)$
 - $\forall \varepsilon > 0 \exists \kappa = \kappa(\varepsilon) > 0$ such that if $x > \kappa$ then $l - \varepsilon < f(x) < l + \varepsilon$
- $-\infty =$ given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with A unbounded above $\lim_{x \rightarrow +\infty} f(x) = -\infty$ means:
 - \forall neighborhood $\mathcal{U}(-\infty) \exists$ a neighborhood $\mathcal{V}(-\infty)$ such that if $x \in \mathcal{V}(-\infty)$ then $f(x) \in \mathcal{U}(-\infty)$
 - $\forall \kappa > 0, k > 0, (-\infty, k], f(x) > \kappa$
- $+\infty =$ given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with A unbounded above $\lim_{x \rightarrow +\infty} f(x) = +\infty$ means:
 - \forall neighborhood $\mathcal{U}(+\infty) \exists$ neighborhood $\mathcal{V}(+\infty)$ such that if $x \in \mathcal{V}(+\infty)$ then $f(x) \in \mathcal{U}(+\infty)$
 - $\forall \kappa > 0, k > 0, [k, +\infty), f(x) > \kappa$

- $\lim_{x \rightarrow -\infty} f(x) = l, +\infty, -\infty (l \in \bar{\mathbb{R}}) \rightarrow$

- $l =$ given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with A unbounded above $\lim_{x \rightarrow -\infty} f(x) = l$ means:
 - \forall neighborhood $\mathcal{U}(l) \exists$ neighborhood $\mathcal{V}(-\infty)$ such that if $x \in \mathcal{V}(-\infty)$ then $f(x) \in \mathcal{U}(l)$
 -
- $-\infty =$ given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with A unbounded above $\lim_{x \rightarrow -\infty} f(x) = -\infty$ means:
 - \forall neighborhood $\mathcal{U}(-\infty) \exists$ neighborhood $\mathcal{V}(-\infty)$ such that if $x \in \mathcal{V}(-\infty)$ then $f(x) \in \mathcal{U}(-\infty)$
 -
- $+\infty =$ given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with A unbounded above $\lim_{x \rightarrow -\infty} f(x) = +\infty$ means:
 - \forall neighborhood $\mathcal{U}(+\infty) \exists$ neighborhood $\mathcal{V}(+\infty)$ such that if $x \in \mathcal{V}(+\infty)$ then $f(x) \in \mathcal{U}(+\infty)$
 -

LIMITS FROM BELOW / ABOVE

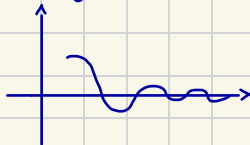
- given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with A unbounded below $\lim_{x \rightarrow -\infty} f(x) = l^+ / l^-$ means:
 - $l^- = \forall$ left neighborhood $U^-(l) \exists$ a neighborhood $V(-\infty)$ such that if $x \in V(-\infty)$ then $f(x) \in U^-(l)$
 - $l^+ = \forall$ right neighborhood $U^+(l) \exists$ a neighborhood $V(-\infty)$ such that if $x \in V(-\infty)$ then $f(x) \in U^+(l)$
- given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with A unbounded below $\lim_{x \rightarrow +\infty} f(x) = l^+ / l^-$ means:
 - $l^- = \forall$ left neighborhood $U^-(+\infty) \exists$ a neighborhood $V(+\infty)$ such that if $x \in V(+\infty)$ then $f(x) \in U^-(+\infty)$
 - $l^+ = \forall$ right neighborhood $U^+(+\infty) \exists$ a neighborhood $V(+\infty)$ such that if $x \in V(+\infty)$ then $f(x) \in U^+(+\infty)$
- given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with A unbounded below $\lim_{x \rightarrow c} f(x) = l^+ / l^-$ means:
 - $l^- = \forall$ left neighborhood $U^-(l) \exists$ a neighborhood $V(l)$ such that if $x \in V(l)$ then $f(x) \in U^-(l)$
 - $l^+ = \forall$ right neighborhood $U^+(l) \exists$ a neighborhood $V(l)$ such that if $x \in V(l)$ then $f(x) \in U^+(l)$
- given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with A unbounded below $\lim_{x \rightarrow c^-} f(x) = l^+ / l^-$ means:
 - $l^- = \forall$ left neighborhood $U^-(l^-) \exists$ a neighborhood $V(l^-)$ such that if $x \in V(l^-)$ then $f(x) \in U^-(l^-)$
 - $l^+ = \forall$ right neighborhood $U^+(l^-) \exists$ a neighborhood $V(l^-)$ such that if $x \in V(l^-)$ then $f(x) \in U^+(l^-)$
- given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with A unbounded below $\lim_{x \rightarrow c^+} f(x) = l^+ / l^-$ means:
 - $l^- = \forall$ left neighborhood $U^-(l^+) \exists$ a neighborhood $V(l^+)$ such that if $x \in V(l^+)$ then $f(x) \in U^-(l^+)$
 - $l^+ = \forall$ right neighborhood $U^+(l^+) \exists$ a neighborhood $V(l^+)$ such that if $x \in V(l^+)$ then $f(x) \in U^+(l^+)$

HORIZONTAL ASYMPTOTES

- the straight line $y=l$ is a horizontal asymptotes for $y=f(x)$ when $x \rightarrow +\infty$ if $\lim_{x \rightarrow +\infty} f(x) = l$

- a function can cross a horizontal asymptote (even infinitely many times)

ex. $y = \frac{\sin x}{x} \rightarrow \lim_{x \rightarrow +\infty} \frac{\sin x}{x} = 0$ $\nearrow [-1, 1]$



- when is it that $\lim \neq$?

① at $+\infty \rightarrow$

- $\lim_{x \rightarrow +\infty} \sin x \neq$
- $\lim_{x \rightarrow +\infty} \tan x \neq$

② at $x=c \rightarrow$

- $\lim_{x \rightarrow 0} \frac{1}{x} \neq$

CHARACTERIZATION OF LIMITS OF FUNCTIONS WITH LIMITS OF SEQUENCES

given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$

given $c \in \mathbb{R}$, limit point of A (also $+\infty$ and $-\infty$)

then: $\rightarrow = l, +\infty, -\infty$

$$\lim_{x \rightarrow c} f(x) = L \in \bar{\mathbb{R}} \iff \forall \text{ sequence } x_n \text{ chosen in } A \text{ with } x_n \neq c \forall n, x_n \rightarrow c \implies f(x_n) \rightarrow L$$

UNIQUENESS OF THE LIMIT

given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$

given $c \in \mathbb{R}$ limit point of A (also $+\infty$ and $-\infty$)

if $\lim_{x \rightarrow c} f(x) \exists$ then it is unique

PERSISTENCE OF SIGN

consider $f(x)$ defined in a neighborhood of c at most excluded c (that is: $U(c) \setminus \{c\}$)

suppose $f(x) \geq 0$ in $U(c) \setminus \{c\}$

suppose $\exists \lim_{x \rightarrow c} f(x)$, then also $\lim_{x \rightarrow c} f(x) \geq 0$

OPERATIONS WITH LIMITS

consider $f(x), g(x)$ such that $\exists \lim_{x \rightarrow c} f(x) = L, \lim_{x \rightarrow c} g(x) = \gamma$ with $L, \gamma \in \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ then:

$$\textcircled{1} \lim_{x \rightarrow c} (f(x) + g(x)) = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$$

$$\textcircled{2} \lim_{x \rightarrow c} (f(x) \cdot g(x)) = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x)$$

$$\textcircled{3} \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$$

provided that there is no indeterminate form: $+\infty - \infty, \frac{\infty}{\infty}, 0 \cdot \infty, \frac{0}{0}$

COMPARISON CRITERION

suppose:

- ① $f(x), g(x), h(x)$ defined in $U(c) - \{c\}$
- ② $f(x) \leq g(x) \leq h(x) \quad \forall x \in U(c) - \{c\}$
- ③ $\lim_{x \rightarrow c} f(x) = l, \lim_{x \rightarrow c} h(x) = l \rightarrow \lim_{x \rightarrow c} g(x) = l$

INDETERMINATE FORMS

if $x \rightarrow +\infty$ everything is exactly like with sequences \rightarrow

$$\text{ex. } \lim_{x \rightarrow +\infty} \frac{x^3 + e^{-x} + 5}{x^4 + e^{\frac{1}{x}} + \sin x} \rightarrow 0 = \lim_{x \rightarrow +\infty} \frac{x^3}{x^4} = 0^+$$

\downarrow \downarrow
 1 [-1, 1]

CHANGE OF VARIABLE

$$\text{ex. } \lim_{x \rightarrow 0^+} x \ln x = (\text{change variable } \rightarrow t = \frac{1}{x} \text{ and } x = \frac{1}{t}) \lim_{t \rightarrow +\infty} \frac{1}{t} \ln \frac{1}{t} = \lim_{t \rightarrow +\infty} \frac{1}{t} \ln(t^{-1})$$

$$= \lim_{t \rightarrow +\infty} -\frac{\ln t}{t} = 0^-$$

\sim, O FOR FUNCTIONS

- $f(x) \sim g(x)$ as $x \rightarrow c$ if $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = 1$
- $f(x) = o[g(x)]$ as $x \rightarrow c$ if $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = 0$

FUNDAMENTAL LIMITS

$$\textcircled{1} \lim_{x \rightarrow +\infty} \left(1 + \frac{a}{x}\right)^x = e^a$$

$$\textcircled{2} \lim_{x \rightarrow 0} (1 + ax)^{\frac{1}{x}} = e^a$$

$$\textcircled{3} \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \rightarrow \text{also: } e^x - 1 \sim x \text{ as } x \rightarrow 0$$

$$e^x \sim 1 + x \text{ as } x \rightarrow 0$$

$$\textcircled{4} \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 \left(\frac{0}{0}\right) \rightarrow \text{also: } \ln(1+x) \sim x \text{ as } x \rightarrow 0$$

$$\textcircled{5} \lim_{x \rightarrow 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \left(\frac{0}{0}\right) \rightarrow \text{also: } (1+x)^\alpha - 1 \sim \alpha x \text{ as } x \rightarrow 0$$

(in particular)

$$\textcircled{6} \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \left(\frac{0}{0}\right) \rightarrow \text{also: } \sin x \sim x \text{ as } x \rightarrow 0$$

$$\left(\alpha = \frac{1}{2}\right) \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \frac{1}{2}$$

$$\rightarrow \text{also: } \sqrt{1+x} - 1 \sim \frac{1}{2} x$$

$$\textcircled{7} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2} \left(\frac{0}{0}\right) \rightarrow \text{also: } 1 - \cos x \sim \frac{1}{2} x^2 \text{ as } x \rightarrow 0$$

$$\cos x \sim 1 - \frac{1}{2} x^2 \text{ as } x \rightarrow 0$$

$$\sqrt{1+x} \sim 1 + \frac{1}{2} x$$

CONTINUITY

CONTINUITY AT x_0

given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, given a point $x_0 \in A$, then:

- if x_0 is a limit point of A , we say that f is continuous at x_0 if:
 - $\exists \lim_{x \rightarrow x_0} f(x) = l$
 - $l = f(x_0)$
- if x_0 is an isolated point of A we just say that f is continuous at x_0 (this def. needs to more general results)

CONTINUITY AT A

consider $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, we say that f is continuous in A if f is continuous $\forall x_0 \in A$

(at a boundary point continuity means continuity from the side for which the function exists \rightarrow only on the right/left)

REMARK = the graph of f is made of two pieces because the domain A is made of two pieces, in fact:

- if the domain A is an interval then it is not continuous
- if the domain A is not an interval then it is continuous

DISCONTINUITIES

when f is not continuous at $x_0 \in A$, we say x_0 is a point of discontinuity

- ① hole (eliminable discontinuity) \rightarrow
 - $\exists \lim_{x \rightarrow x_0} f(x) = l$
 - $l \neq f(x_0)$
- ② jump (discontinuity of I kind) \rightarrow
 - $\exists \lim_{x \rightarrow x_0^-} f(x) = l_1$ (finite)
 - $\exists \lim_{x \rightarrow x_0^+} f(x) = l_2$ (finite)
 - $l_1 \neq l_2$

(jumps cannot be eliminated)
- ③ all others (discontinuities of II kind, essential discontinuities) \rightarrow at least one of the two \pm -sided limits \nexists or is an infinity

CHARACTERIZATION OF CONTINUITY WITH LIMITS OF SEQUENCES

given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, given $x_0 \in A$, then:

f is continuous at $x_0 \iff \forall$ sequence x_n of points of A , $x_n \rightarrow x_0 \implies f(x_n) \rightarrow f(x_0)$

CONTINUITY OF ELEMENTARY FUNCTIONS

every elementary function is continuous in its natural domain A

CONTINUITY AND OPERATIONS

- consider $f, g: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, continuous, then:

- $f+g$ = continuous
 - fg = continuous
 - $\frac{f}{g}$ = continuous
- at all points at which $g(x) \neq 0$

- consider $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $g: B \subseteq \mathbb{R} \rightarrow \mathbb{R}$ such that $g \circ f$ is defined \rightarrow if f, g are continuous then $g \circ f$ is continuous

INVERTIBILITY / STRICT MONOTONICITY

suppose $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$

suppose A is an interval and f is continuous in A , then:

f invertible in $A \iff f$ strictly monotone in A

WEIERSTRASS'S THEOREM

suppose $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, if:

① A is a closed and bounded set in \mathbb{R}

② f is continuous in A

then:

$\exists \max f(x) = M, \exists \min f(x) = m$ in A (global)

BALZANO'S THEOREM / ZERO-VALUE THEOREM

suppose $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, if:

① A is a closed and bounded interval $[a, b]$

② f is continuous in A

③ $f(a) \cdot f(b) \leq 0$

then:

$\exists c \in [a, b]: f(c) = 0$

moreover: if f is strictly monotonic then this point c is unique

DARBOUX'S THEOREM / INTERMEDIATE-VALUE THEOREM

take $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, if:

① A is a closed and bounded interval $[a, b]$

② f is continuous in A

then:

$\forall L: \min f(x) \leq L \leq \max f(x) \exists c: a \leq c \leq b$ with $f(c) = L$

if f is continuous on a closed and bounded interval $[a, b]$, then $f([a, b])$ is a closed and bounded interval $[m, x]$

also:

$[a, b]$ closed and bounded interval = compact and convex set in \mathbb{R}

→

$f([a, b])$ closed and bounded interval = compact and convex set in \mathbb{R}

LIMITS AND CONTINUITY FOR FUNCTIONS $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$

consider a function $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}: y = f(x_1, x_2, \dots, x_n)$

usually we have a function $f: A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}: y = f(x_1, x_2)$

LIMITS

given a function $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ given ξ limit point of A saying that $\lim_{x \rightarrow \xi} f(x) = l$, $+\infty, -\infty$ means that: \forall neighborhood $U(l, +\infty, -\infty) \exists$ a neighborhood $V(\xi)$ such that $\forall x \in V(\xi), x \neq \xi, f(x) \in U(l, +\infty, -\infty)$

CONTINUITY AT x_0

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, x_0 \in A$

① if x_0 is a limit point of A we say f is continuous at x_0 if:

- $\exists \lim_{x \rightarrow x_0} f(x) = l$
- $l = f(x_0)$

② if x_0 is an isolated point of A we just say that f is continuous at x_0

CONTINUOUS WA

given $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, f is continuous in A if f is continuous $\forall x_0 \in A$

THEOREMS ON CONTINUITY

CONTINUITY OF ELEMENTARY FUNCTIONS

every elementary function $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous in its natural domain A

WEIERSTRASS'S THEOREM

consider a function $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, if:

- ① A is a closed and bounded set (compact) in \mathbb{R}^n
- ② f is continuous in A

then:

$\exists \max f(x) = x, \exists \min f(x) = m$ in A

if $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous on a compact and convex set $A \subseteq \mathbb{R}^n$, then $f(A)$ is a closed and bounded interval $[m, x]$

compact and convex set in $\mathbb{R}^n \rightarrow$ compact and convex set in \mathbb{R}

COERCIVITY

a function $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be coercive on a subset $c \subseteq A$ if $\exists t \in \mathbb{R}$ such that the set: $(f \geq t) \cap c = \{(x_1, \dots, x_n) \in c : f(x_1, \dots, x_n) \geq t\}$ is non-empty and compact

that is, at least one upper contour set has a non-empty and compact intersection with c

when $c = A$ the function is just said to be coercive with no specification (coercivity is a joint property of the function f and the set c)

PRESERVATION OF COERCIVITY FOR CONTINUOUS FUNCTIONS WITH RESPECT TO STRICTLY INCREASING COMPOSITIONS

consider $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, continuous on A

consider $g: B \subseteq \mathbb{R} \rightarrow \mathbb{R}$, strictly increasing on B and such that $f(A) \subseteq B$

then: if f is coercive on $c \subseteq A$, also $g \circ f$ is coercive on $c \subseteq A$

THEOREM

if $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous on a compact subset $c \subseteq A$ then f is coercive on c (and actually, whenever $(f \geq t)$ is non-empty, all the sets: $(f \geq t) \cap c$ are compact and non-empty)

TOLELLI'S THEOREM

consider a function $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ which is both coercive and continuous on a subset $c \subseteq A$

then the function has a maximizer in c that is $\exists \hat{x} \in c$ such that: $f(\hat{x}) = \max_{x \in c} f(x)$

SUPERCOERCIVITY

a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be supercoercive in \mathbb{R}^n if: \forall sequence $x_n, x_n \in \mathbb{R}^n$, $\|x_n\| \rightarrow +\infty \Rightarrow f(x_n) \rightarrow -\infty$ (that is: whenever x_n goes indefinitely far from o , $f(x_n)$ goes down to $-\infty$)

THEOREM I

a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is supercoercive in $\mathbb{R}^n \iff$ all its upper contour sets are bounded

THEOREM II

consider a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$

if f is supercoercive in \mathbb{R}^n and continuous on a closed subset $c \subseteq \mathbb{R}^n$ then it is coercive on $c \rightarrow$ all the sets $(f \geq t) \cap c$ are compact

**theorems and proofs for
limits of functions and
continuity**

THEOREM OF THE UNIQUENESS OF THE LIMIT

given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$

given $x_0 \in \mathbb{R}$ limit point of A

if $\lim_{x \rightarrow x_0} f(x) \exists$ then it is unique

PROOF

given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $x_0 \in \mathbb{R}$ limit point of A

$\exists! l \in \bar{\mathbb{R}}$ such that $\lim_{x \rightarrow x_0} f(x) = l$

suppose $l_1 \neq l_2$

suppose $\{x_n\}$ sequence in A with $x_n \neq x_0$ such that $x_n \rightarrow x_0$

$\lim_{x \rightarrow x_0} f(x) = l \iff x_n \rightarrow x_0$ then $f(x_n) \rightarrow l \implies f(x_n) \rightarrow l$, and

$f(x_n) \rightarrow l_2 =$ contradiction to the uniqueness of limits for sequences = absurd

COMPARISON CRITERION

suppose

- ① $f(x), g(x), h(x)$ defined in $\mathcal{D}(c) \setminus \{c\}$
- ② $f(x) \leq g(x) \leq h(x) \quad \forall x \in \mathcal{D}(c) \setminus \{c\}$
- ③ $\lim_{x \rightarrow c} f(x) = l \quad \lim_{x \rightarrow c} h(x) = l \rightarrow \lim_{x \rightarrow c} g(x) = l$

PROOF

we know that $f(x) \leq g(x) \leq h(x)$

$$\lim_{x \rightarrow c} f(x) = l \rightarrow \forall \varepsilon > 0 \exists \bar{n}_1 \in \mathbb{N} : \forall n \geq \bar{n}_1 \quad l - \varepsilon < f(x) < l + \varepsilon$$

$$\lim_{x \rightarrow c} h(x) = l \rightarrow \forall \varepsilon > 0 \exists \bar{n}_2 \in \mathbb{N} : \forall n \geq \bar{n}_2 \quad l - \varepsilon < h(x) < l + \varepsilon$$

$$\text{take } n = \max(\bar{n}_1, \bar{n}_2, \bar{n}_3) \rightarrow l - \varepsilon < f(x) \leq g(x) \leq h(x) < l + \varepsilon$$

$$\downarrow$$
$$l - \varepsilon < g(x) < l + \varepsilon$$

$$\lim_{x \rightarrow c} g(x) = l$$

LIMIT OF A SUM OF FUNCTIONS

consider $f(x), g(x)$ such that $\exists \lim_{x \rightarrow c} f(x) = L$ $\lim_{x \rightarrow c} g(x) = \varkappa$ with $L, \varkappa \in \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$

then:

$$\textcircled{1} \lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$$

provided that there is

$$\textcircled{2} \lim_{x \rightarrow c} [f(x) \cdot g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x)$$

no indeterminate form

$$\textcircled{3} \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$$

PROOF

take any sequence x_n of points of A with $x_n \neq c \forall n$ and $x_n \rightarrow c$

using the comparison criterion theorem it must be that $f(x_n) \rightarrow L$ and

$g(x_n) \rightarrow \varkappa$

if $L + \varkappa$ is not an indeterminate form using the theorem on the limit of the sum of sequences we have that $(f+g)(x_n) \rightarrow L + \varkappa$

since this is true \forall such sequence x_n by the comparison criterion theorem this

means that $\lim_{x \rightarrow c} [f(x) + g(x)] = L + \varkappa$

CHARACTERIZATION OF CONTINUITY WITH LIMITS OF SEQUENCES

given $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$

given $x_0 \in A$, then f is continuous at $x_0 \Leftrightarrow \forall$ sequence x_n of points of A ,
 $x_n \rightarrow x_0 \Rightarrow f(x_n) \rightarrow f(x_0)$

PROOF

① x_0 limit point

$$\forall n: x_n \rightarrow x_0 \Rightarrow f(x_n) \rightarrow f(x_0) \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = f(x_0)$$

it follows from theorem on the characterization of limits of functions with limits of sequences

② x_0 isolated point

then by definition $f(x)$ is continuous

the only sequence x_n is $x_n = x_0, x_0, x_0, x_0, x_0, x_0$

and the corresponding sequence of the image is the constant sequence:

$$f(x_n) = f(x_0), f(x_0), f(x_0)$$

TOLELLI

consider a function $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ which is both coercive and continuous on a subset $C \subseteq A$

then the function has a maximizer in C that is $\exists x_0 \in C$ such that: $f(x_0) = \max_{x \in C} f(x)$

PROOF

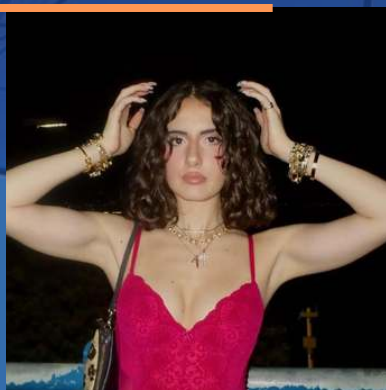
since f is coercive on C , $\exists t \in \mathbb{R}$ such that $(f \geq t) \cap C \neq \emptyset$ and compact

$\Gamma = (f \geq t) \cap C$ such that f is continuous on $\Gamma \Rightarrow$ for the Weierstrass's theorem $\exists x_0 \in \Gamma: f(x_0) \geq f(x) \forall x \in \Gamma$

in the points of $C \notin \Gamma$ $f(x) < t$ and $f(x_0) \geq t \Rightarrow$

$f(x_0) \geq f(x) \forall x \in C \Rightarrow f(x_0) = \max_{x \in C} f(x)$

FOR DOUBTS OR SUGGESTIONS ON THE HANDOUTS



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