QUANTITATIVE METHODS (PRIMO PARZIALE)

NOTES

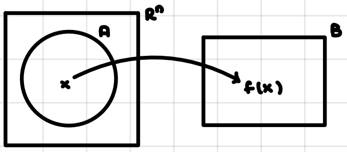
A.Y. 2023 - 2024

A cura di Elena Cacioli e
Albino Trapuzzano

5 febbraio 2024 – lezione 1

Differential calculus

function of several variables :

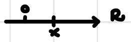


R : codomain

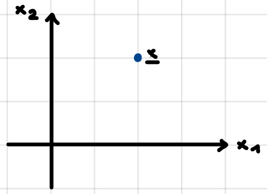
$A \subseteq \mathbb{R}^m$ →

- $f(x_1, x_2, \dots x_n)$
- $(x_1, x_2, \dots x_n) = \underline{x}$

$n=1 \rightarrow x \in \mathbb{R}$



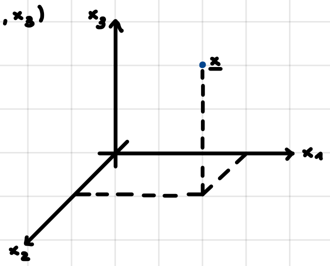
$n=2 \rightarrow \underline{x} = (x_1, x_2) \in \mathbb{R}^2$



x_1, x_2 are the independent variables

$z = f(\underline{x})$ is the (unique) dependent variable

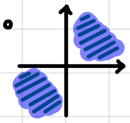
$n=3 \rightarrow \underline{x} = (x_1, x_2, x_3)$



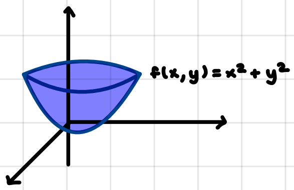
$f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$
↓ domain ↓ codomain

ex :

- B.M.I. → $f(x_1, x_2)$ (x_1 ↓ weight, x_2 ↓ height)
- $f(K, L)$ = production function
- $f(x, y) = 2x + 3y \rightarrow A = \text{dom} f = \mathbb{R}^2$
- $f(x, y) = \ln(xy) \rightarrow x \cdot y > 0$



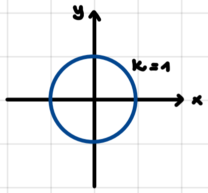
- graph of $f(x,y)$: is a surface in 3D



- contour lines of level K : $f(x,y) = K$ fixed level
  ↳ all points at which f has the same value K → $L_f(K) = \{(x,y) \in A \subseteq \mathbb{R}^2 : f(x,y) = K\}$ level set → line in the plane

ex :

$f(x,y) = x^2 + y^2 = K$ $K=1 \rightarrow x^2 + y^2 = 1$



if I choose $K<0$ = it's impossible

- $f(x) = x^2 + 3x \rightarrow f'(x) = 2x + 3$
- $f(x,y) = x^2 + y^2$ → $f'_x = 2x$ (y^2 is like a number so f' is 0) ; → $f'_y = 2y$ (x^2 is like a number so f' is 0) → partial derivative

↓

A function $f(x_1, x_2 \dots x_n)$ has (may have) n partial derivatives

→ $f'_{x_1}, f'_{x_2} \dots f'_{x_n} = \dfrac{\partial f}{\partial x_n}$ → gradient vector :

$\nabla f = [f'_{x_1} \dots f'_{x_n}]$
↓
nabla

→ $\nabla f = [2x \quad 2y]$

6 febbraio 2024 – lezione 2

Second order partial derivatives

- n=1 → $f(x), f'(x), f''(x)$
- $n>1 \rightarrow f(\underline{x}), \nabla f(\underline{x}), ?$ → n=2 : $f(x,y) \rightarrow \nabla f(x,y) = [f_x(x,y) \quad f_y(x,y)]$

→ $\dfrac{\partial f_x}{\partial x} = \dfrac{\partial^2 f}{\partial x^2}$ or $f''_{x,x}$

→ $\dfrac{\partial f_x}{\partial y} = \dfrac{\partial^2 f}{\partial x \cdot \partial y}$ or $f''_{x,y}$

→ $\dfrac{\partial f_y}{\partial x} = \dfrac{\partial^2 f}{\partial x \cdot \partial y}$ or $f''_{x,y}$

→ $\dfrac{\partial f_y}{\partial y} = \dfrac{\partial^2 f}{\partial y^2}$ or $f''_{y,y}$

ex.

- $f(x,y) = x^2 + y^2 \rightarrow \nabla f = [\overset{f_x \downarrow}{2x} \quad \overset{f_y \downarrow}{2y}]$

$$\rightarrow \underbrace{\frac{\partial^2 f}{\partial x^2} = 2, \frac{\partial^2 f}{\partial x \cdot \partial y} = 0, \frac{\partial^2 f}{\partial y \cdot \partial x} = 0, \frac{\partial^2 f}{\partial y^2} = 2}$$

↓

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2 \cdot 2 \text{ matrix}$$

hessian matrix : simmetric

in most cases
↓
Schwartz system

- $f(x,y) = e^x + 2x \cdot y \rightarrow \nabla f = [e^x + 2y \quad 2x]$

$$\rightarrow \underbrace{\frac{\partial^2 f}{\partial x^2} = e^x, \frac{\partial^2 f}{\partial x \cdot \partial y} = 2, \frac{\partial^2 f}{\partial y \cdot \partial x} = 0, \frac{\partial^2 f}{\partial y^2} = 2}$$

$$\Big\downarrow$$

$$\begin{bmatrix} e^x & 2 \\ 2 & 0 \end{bmatrix}$$

- **n=3** : $\nabla f = [f'_{x_1}, f'_{x_2}, f'_{x_3}]$

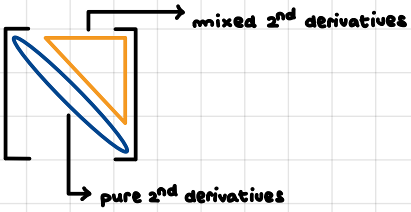
$$\hookrightarrow \nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \cdot \partial x_1} & \frac{\partial^2 f}{\partial x_3 \cdot \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \cdot \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \cdot \partial x_3} \\ \frac{\partial^2 f}{\partial x_1 \cdot \partial x_3} & \frac{\partial^2 f}{\partial x_3 \cdot \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} \longrightarrow \text{ex. } f(x,y,z) = x^2 + y^2 + 3z^2$$

$\nabla f = [2x \quad 2y \quad 6z]$

$$\hookrightarrow \nabla^2 f = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

in general :

- $\nabla^2 f$ is a n·n matrix
- $\nabla^2 f$ is simmetric → does not matter the order of derivation
- the derivatives $\frac{\partial^2 f}{\partial x_k^2}$ are said **pure** → they are on the diagonal
- the derivatives $\frac{\partial^2 f}{\partial x_k \cdot \partial x_s}$ with k=s are said **mixed** → outside the diagonal ⟶ $\nabla^2 f =$



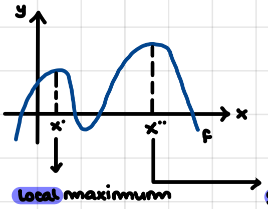
Unconstrained optimization

- $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ **objective function**
- $\underset{\underline{x} \in A}{\text{MAX}} f(\underline{x})$ or $\underset{\underline{x} \in A}{\text{MIN}} f(\underline{x})$ ⟶ **optimization problem** : on the domain A of f

↓

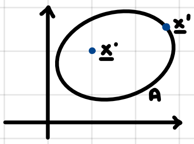
how to solve an optimization problem :

n=1



local maximum point

global maximum point : we have to consider **interior** points of domain A



$\underline{x}^{\cdot}$ interior

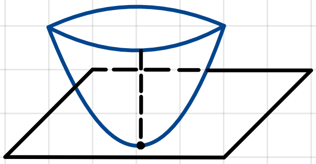
$\underline{x}'$ **boundary**

first order condition (foc) = if :

NB : $\underline{x}^{\cdot}$ optimum point

⟶ $\nabla f(\underline{x}^{\cdot}) = 0$

- $\underline{x}^{\cdot}$ interior point of A at which f admits ∇f
- $\underline{x}^{\cdot}$ is a MAX (MIN) point then → **stationary point** (necessary condition) $\nabla f(\underline{x}^{\cdot}) = 0$



8 febbraio 2024 - lezione 3

Max and min continued

$f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ diferentiable

ex (FOC):

- $f(x,y) = x^2 + y^2 \rightarrow \operatorname{dom} f = \mathbb{R}^2 \rightarrow$ the first order derivatives are:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x = 0 \\ \dfrac{\partial f}{\partial y} = 2y = 0 \end{cases} \rightarrow \begin{cases} x^{\cdot} = 0 \\ y^{\cdot} = 0 \end{cases} \rightarrow (x^{\cdot}, y^{\cdot}) = (0,0) \text{ is the unique}$$

stationary point

$\rightarrow$ if there is an max/min point of f on $\mathbb{R}^2$

it is the origin

! if no stationary point for

$f: \mathbb{R}^n \rightarrow \mathbb{R}$ differentiable $\longrightarrow$ no optima for $f: \mathbb{R}^n \rightarrow \mathbb{R}$

Second order condition (II test):

- $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ twice differentiable at every interior point of A
- the North-West principal minor of the hessian matrix $\nabla^2 f(\underline{x}^{\cdot})$ are involved:

$$n=2 \rightarrow \nabla^2 f = \begin{bmatrix} f'_{xx} & f'_{xy} \\ f'_{yx} & f'_{yy} \end{bmatrix} \rightarrow \begin{cases} H_1 = f'_{xx} \\ H_2 = \det \nabla^2 f \end{cases}$$

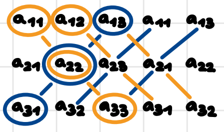
$$n=3 \rightarrow \nabla^2 f = \begin{bmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{zz} \end{bmatrix} \rightarrow \begin{cases} H_1 = f''_{xx} \\ H2 = \det \begin{matrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{matrix} \\ H_3 = \det \nabla^2 f \end{cases}$$

$\Longrightarrow$

Remark:

- if the sign rules holds everywhere then the max/min point is global
- if at least one sign is wrong $[\underline{x}^{\cdot} < 0]$ then $\underline{x}^{\cdot}$ is a saddle point in most cases
- in case some $H_k = 0$ (and no minor has wrong signs) then nothing can be said about the nature of $\underline{x}^{\cdot}$

$\rightarrow$ on demand: $\rightarrow$ Sarrus rule for det $\underset{3\cdot 3}{A}$



$\rightarrow$ det A = orange − blue

theorem:

if $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is twice differentiable at the stationary point $\underline{x}^{\cdot}$ $(\nabla f(\underline{x}^{\cdot}) = \underline{0})$ which is interior to A then:

- if all the NW principal minors of $\nabla^2 f(\underline{x}^{\cdot})$ are positive then $\underline{x}^{\cdot}$ is (at least) a local minum
- if the NW principal minors are $H_1 < 0, H_2 > 0, H_3 < 0 \ldots$ then $\underline{x}^{\cdot}$ is (at least) local maximum point

ex.

$f(x,y) = x^2 + y^2 \rightarrow \nabla f = [2x \quad 2y] \rightarrow (x^{\cdot}, y^{\cdot}) = (0,0)$

- $\nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \forall (x,y) \in \mathbb{R}^2 \rightarrow H_1 = 2 > 0$

$H_2 = \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4 > 0 \rightarrow (0,0)$ minimum point

$\longrightarrow$ ex:

$f(x,y,z) = z \cdot x - x^2 - y^2 - z^2$ max/min? at least local

- Look for stationary point

$$\begin{cases} \dfrac{\partial f}{\partial x} = z - 2x = 0 \\ \dfrac{\partial f}{\partial y} = -2y = 0 \\ \dfrac{\partial f}{\partial z} = x - 2z = 0 \end{cases} \rightarrow \begin{cases} z = 2x \rightarrow x = 0 \\ y = 0 \\ z = 2z \rightarrow z = 0 \end{cases} \rightarrow (0,0,0) \text{ is the unique stationary point}$$

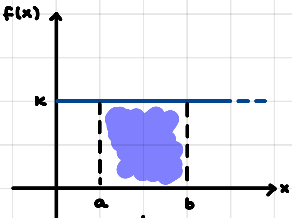
- $\nabla^2 f = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \forall (x,y,z) \in \mathbb{R}^3 \rightarrow H_1 = -2$

$H_2 = \det \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = 4 \longrightarrow (0,0,0)$ is the global max point for f on $\mathbb{R}^3$

$H_3 = \det \nabla^2 f = -6 < 0$

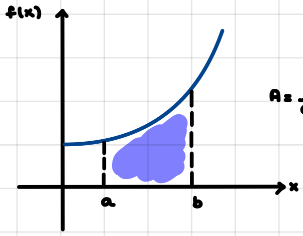
12 febbraio 2024 - lezione 4

Definite integral



$\hookrightarrow A = (b-a) \cdot K$

$\longrightarrow$



$f: \mathbb{R} \rightarrow \mathbb{R}$ positive and bounded on $[a,b]$

$A = \int_a^b f(x)\,dx$

$\hookrightarrow$ definite integral of f on $[a,b]$

$\hookrightarrow \Delta = \dfrac{b-a}{n} : f(c_K), K = 1, \ldots, n$

aproximate A with the plurirectangular area: $S_n = \Delta f(c_1) + \ldots + \Delta f(c_n) = \sum_{K=1}^{n} \Delta f(c_K)$

$n \rightarrow +\infty$

$\hookrightarrow S_n \rightarrow A = \int_a^b f(x)\,dx$

Which functions are Remann integrable on $[a,b]$, that is $\int_a^b f(x)\,dx$ does exist? → necessary condition

- $\exists \int_a^b f(x)\,dx \longrightarrow f$ is bounded on $[a,b]$ ⟶ how to compute $\int_a^b f(x)\,dx$?
- f continuous on $[a,b] \longrightarrow \exists \int_a^b f(x)\,dx$
- f monotone on $[a,b] \longrightarrow \exists \int_a^b f(x)\,dx$

theorem: if $f: \mathbb{R} \to \mathbb{R}$ is Remann integrable on $[a,b]$ and admits an antiderivative $G(x)$ on $[a,b]$ then

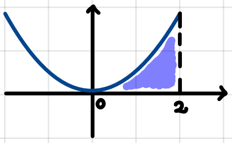
$$\int_a^b f(x)\,dx = G(b) - G(a) = \left[G(x)\right]_a^b$$

⟶ therefore, to compute $\int_a^b f(x)\,dx$, you must:

- compute antiderivative G of f does not matter which one
- compute $G(b)$ and $G(a)$
- compute the difference $G(b)-G(a)$

ex:

- $f(x) = x^2 \longrightarrow \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$



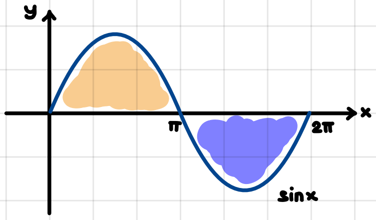
Properties:

- additivity with respect to the integration interval ⟶ for every $c \in [a,b]$ it is $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ ⟶ $f(x) = \begin{cases} x^2 & x \leq 2 \\ 3 & x > 2 \end{cases} \rightarrow \int_0^5 f(x)\,dx = \underbrace{\int_0^2 x^2\,dx}_{\frac{8}{3}} + \int_2^5 3\,dx = \frac{35}{3} \sim$

  $\int_2^5 3\,dx \rightarrow \left[3x\right]_2^5 = 9$

- $\int_a^a f(x)\,dx = 0$ does not matter if you add/eliminate one single point to $[a,b]$
- $\int_b^a f(x)\,dx = G(a) - G(b) = -\int_a^b f(x)\,dx$
- $\int_a^b f(x)\,dx$ Linear with respect to f:
  - additive: $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ for every integrable f, g
  - homogeneous: $\int_a^b \alpha \cdot f(x)\,dx = \alpha \int_a^b f(x)\,dx$ for every $\alpha \in \mathbb{R}$ and every -egrable f
  - Linear: $\int_a^b \alpha f(x) + \beta g(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx \longrightarrow \int_0^2 x^2 + 2x + 3\,dx = \left[\frac{x^3}{3} + x^2 + 3x\right]_0^2 = \frac{8}{3} + 4 + 6 = \frac{38}{3}$

- monotone increasing: $0 \leq f(x) \leq g(x)$ for all $x \in [a,b]$
  $\rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$
  - $f(x) > 0 \ \forall x \in [a,b] \rightarrow \int_a^b f(x)\,dx > 0$
  - $f(x) < 0 \ \forall x \in [a,b] \rightarrow \int_a^b f(x)\,dx < 0 \rightarrow A = -\int_a^b f(x)\,dx$

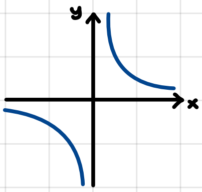
- 

$\int_0^{2\pi} \sin x\,dx = 0$

↳ total area: $A = \int_0^{2\pi} \sin x\,dx = 2A$

ex.

- $f(x) = \frac{1}{x}$ $x \neq 0$



$\rightarrow \int_1^e \frac{1}{x}\,dx = \left[\ln x\right]_1^e = 1$

- $\int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e - 1$

13 Febbraio 2024 - Lezione 5

Computing and antiderivative

Antiderivative of $f: A \subseteq R \to R$ on an interval I is a function G which is derivable on I with $G'(x) = f(x)\ \forall x \in I$ → if G is an antiderivative of f on I then also $P(x) = G(x) + c$ is an antiderivative of f on I → the set of all antiderivatives is called indefinite integral on denoted $\int f(x)\,dx = G(x) + c$

$c \in R$

- elementary functions:

$f(x)$	K	x^α	$\frac{1}{x}$	e^x	a^x	$\sin x$	$\cos x$
$\int f(x)\,dx$	Kx	$\frac{x^{\alpha+1}}{\alpha+1}$	$\text{lm}\lvert x\rvert$	e^x	$\frac{a^x}{\text{lm}a}$	$-\cos x$	$\text{sim}x$

- decomposition: $\int [\alpha \cdot f(x) + \beta \cdot g(x)]\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx \longrightarrow$ ex. $\int \text{sem}x - 5e^x + x^3\,dx = -\cos x - 5e^x + \frac{x^4}{4} + c$
- substitution (change of variable): $\int g[\underset{y=f(x)}{f(x)}] \cdot \underset{dy}{f'(x)\,dx} = \int g(y)\,dy \longrightarrow$ ex. $\int e^{\overset{y=x^2}{x^2}} \cdot \underset{dy}{2x\,dx} = \int e^y\,dy = e^y + c = e^{x^2} + c \longrightarrow \int e^{f(x)} \cdot f'(x)\,dx = e^{f(x)} + c$

ex. $\int \frac{2x}{x^2+1}\,dx = \text{lm}|x^2+1| + c \longrightarrow \int \frac{f'(x)}{f(x)}\,dx = \text{lm}|f(x)| + c$

- remark: $\int_a^b g[f(x)] \cdot f'(x)\,dx = \int_{f(a)}^{f(b)} g(y)\,dy \longrightarrow$ ex. $\int_{0 \to f(0)=1}^{1 \to f(1)=2} \frac{2x}{\underset{y}{x^2+1}}\,dx = [\text{lm}\,y]_1^2 = \text{lm}2$
- by parts: $\int f(x) \cdot g(x)\,dx = f(x) \cdot \int g(x)\,dx - \int \left(\int g(x)\,dx\right) \cdot f'(x)\,dx$ → $f = f'(x)$ finite factor → $g = \int g\,dx$ differential factor
  - ex. $\int x \cdot \text{lm}x\,dx = \text{lm}x \cdot \int x\,dx - \int \left(\int x\,dx\right) \cdot \frac{1}{x}\,dx = \frac{x^2 \cdot \text{lm}x}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx + c$ → $\frac{x^2}{4}$ $= \frac{x^2 \cdot \text{lm}x}{2} - \frac{x^2}{4} + c$
  - ex. $\int x \cdot e^x\,dx = x \cdot \int e^x\,dx - \int \left(\int e^x\,dx\right) \cdot 1\,dx = e^x(x-1) + c$
  - ex. $\int \text{lm}\,x\,dx = \text{lm} \cdot \int 1\,dx - \int \left(\int 1\,dx\right) \cdot \frac{1}{x}\,dx = x(\text{lm}-1) + c$

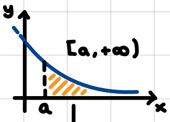
exercises:

- $\int \frac{1}{x(x+1)}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx = \text{lm}|x| - \text{lm}|x+1| + c$
- $\int \text{tg}x\,dx = -\text{lm}|\cos x| + c$

15 Febbraio 2024 - Lezione 6

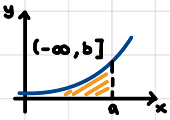
Generalized improper integrals

Reimann integral → f bounded → unbounded intervals :
on
[a,b] bounded

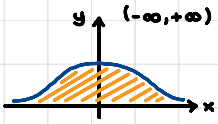


$\int_a^{+\infty} f(x)\,dx$

→ area may be finite

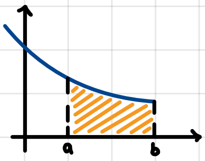


$\int_{-\infty}^{a} f(x)\,dx$

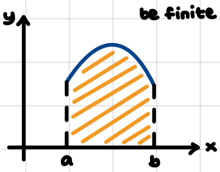


$\int_{-\infty}^{+\infty} f(x)\,dx$

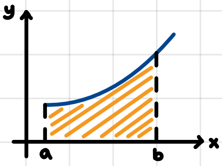
trivial cases:



$\int_a^{+\infty} f(x)\,dx = \int_a^b f(x)\,dx + \int_b^{+\infty} 0\,dx$



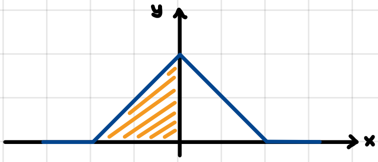
$\int_{-\infty}^{+\infty} f(x)\,dx = \int_a^b f(x)\,dx$



$\int_{-\infty}^{b} f(x)\,dx = \int_{-\infty}^{a} 0\,dx + \int_a^b f(x)\,dx$

ex.

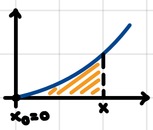
$$f(x) = \begin{cases} x+1 & \text{if } -1 \leq x < 0 \\ 1-x & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x \notin [-1,1] \end{cases}$$



$\int_{-\infty}^{0} f(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} (x+1)\,dx = \frac{1}{2}$

Integral function

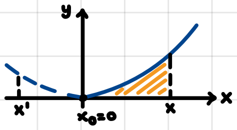
- f integrable on $[a,b] \in R$
- the integral function of f centered at $x_0 \in [a,b]$ → $F(x) = \int_{x_0}^{x} f(t)\,dt$, $x \in [a,b]$



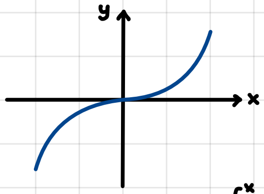
→ $A(x) = \int_0^x f(t)\,dt = F(x)$

ex.

- $F(x)=\int_0^x t^2\,dt=\left[\frac{1}{3}t^3\right]_0^x=\frac{x^3}{3}$ dom f = R



the graph of $F(x)=\int_0^x f(t)\,dt$ is



if f has antiderivative P on [a,b]

→ then the integral function $F(x)=\left[P(x)\right]_{x_0}^x=P(x)-P(x_0)$ → [the one that take 0 values at x_0]

$P(x_0)\downarrow C$

F is one of the antiderivative of f

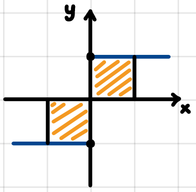
↓

note that is always $F(x)=\int_{x_0}^x f(t)\,dt \rightarrow F(x_0)=\int_{x_0}^x f(t)\,dt=0$

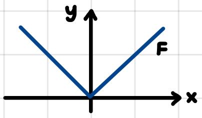
if f is continuous then f admits antiderivative and is integrable too → if f is discontinuous then f can't have antiderivative

ex.

$$f(x)=\begin{cases}-1 & \text{if } x<0\\ 1 & \text{if } x\geq 0\end{cases} \rightarrow F(x)=\int_0^x f(t)\,dt$$



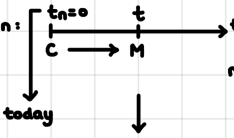
$$F(x)=\begin{cases}\int_0^x -1\,dt & x<0\\ \int_0^x 1\,dt & x\geq 0\end{cases} \rightarrow \begin{cases}-x & x<0\\ x & x\geq 0\end{cases}$$



19 Febbraio 2024 - lezione 7

Financial calculus

- Money and time : leave in bank or invest them → accomunation :



time (year)

money (euros)

- C = principal , is the initial avaiability of money
- M = future value of C after $t-t_0=t-0=t$ years ↳ lifetime of the accomulation process

↓ N.B. : t is
- future date
- lifetime if $t_0=0 \rightarrow t-t_0=t$

↓ $\boxed{I=M-C}$ interest on C for t years

↳ $M=C+I$ additive

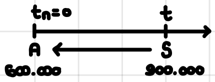
↳ $\boxed{f=\frac{M}{C}}$ accomulation factor after t years

↳ $\boxed{M=C\cdot f}$ multiplicative

ex. : C = 1.000.000 t = 5 years → M = 1.150.000 → risk-free investement

ex.: C = 1.000.000 t = 5 years → M = 1.150.000 →
- I = 150.000 on C after 5 years
- f = 1,15 after 5 years

- Discount :



- S = nominal value
- A = present value or discount value ($t_0=0$)

↓ $\boxed{D=S-A}$ discount

↳ $\boxed{A=S-D}$

↳ $\boxed{\varphi=\frac{A}{S}}$ discount factor

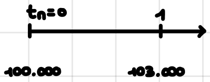
↳ $\boxed{A=S\cdot\varphi}$

→ both f and φ should depend on the lifetime t → $f=f(t)$ and $\varphi=\varphi(t)$

$f=f(t)$ ↓ increasing

$\varphi=\varphi(t)$ ↓ decreasing

ex. :



both accomulation and discount →

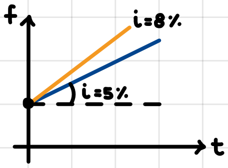
$$\begin{cases}I=3000=D\\ f=1{,}03\\ \varphi=\frac{1}{1{,}03}=\frac{1}{f}\end{cases}$$

Definition: the accomulation factor f and the discount factor φ are said conjugated if $\boxed{f(t)\cdot\varphi(t)=1}$ $t\geq 0$ → some properties of $f(t)\rightarrow\varphi(t)$:

$$\rightarrow\begin{cases}f(t)=\frac{1}{\varphi(t)}\\ \varphi(t)=\frac{1}{f(t)}\end{cases} \text{for all } t\geq 0$$

- $f(0)=1 \rightarrow \varphi(t)=1$
- $f(t)\geq 0 \rightarrow \varphi(t)\geq 0$ for all t
- domain f and φ is $\boxed{I=[0,T]}$
- $f(t)\nearrow \rightarrow \varphi(t)\searrow$

- simple interest : the interest I are proportional to C and t → $\boxed{I=i\cdot C\cdot t \rightarrow M=C\cdot iCt=C\underbrace{(1+it)}_{f(t)}}$ → $i>0$ is the simple annual interest rate



ex.: C = 200.000 t = 1,5 i = 2% simple interest rate per year

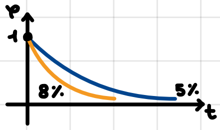
→ $M=200.000\,(1+0{,}02\cdot 1{,}5)=206.000$

20 Febbraio 2024 - lezione 8

Simple (also rational) interest

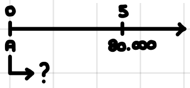
$f(t)=1+it \rightarrow \varphi(t)=\frac{1}{1+it}, t>0$

↳ i = simple (annual) rate of interest



$\frac{1}{1+0{,}05t} > \frac{1}{1+0{,}08t}$ for all $t>0$ → ex: $S=80.000$ $t=5$ years

$i=6\%$ simple



↳ ?

$A=80.000\cdot\frac{1}{1+0{,}06\cdot 5}\sim 61538{,}46$

↓

how many decimal places?:

- money = 2 · rates = 4-6

↓

$S=80.000$ $t=5$ years

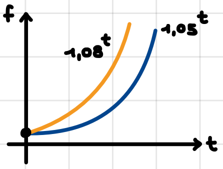
$A=65.000$ $i=?$

$65.000=\frac{80.000}{1+i\cdot 5}$

$i\sim 5\%$

- compound interest:

$f(t)=(1+i)^t$, $i\geq 0$ → compound annual interest rate



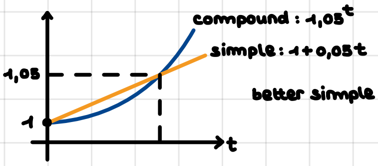
meaning = capitalization of interest also compounding:

- $t=1,2,\dots,n$ years (integer number of years)
- $C=1\xrightarrow{i} m_1=1+i \rightarrow m_2=(1+i)+im_1=(1+i)^2$

$\rightarrow m_n=(1+i)^{n\rightarrow t}$

ex.: $C=100.000$ compound $i=4\%$ $t=3{,}5$ years

$M=100.000\cdot(1+0{,}04)^{3{,}5}\sim 114714{,}07$ → better to invest at the interest rate $i=5\%$ compound or $i=5\%$ simple?



better simple if $t<1$

- compound discount:

$\varphi(t)=\frac{1}{(1+i)^t}=(1+i)^{-t}, t\geq 0$

$i\geq 0$ compound interest rate

ex: $S=10.000$ $t=1{,}5$ years

$i=6\%$ compound → PV? → $A=10.000\cdot 1{,}06^{-1{,}5}\sim 9163{,}07$

→ $D=S-A=836{,}93$

22 Febbraio 2024 - lezione 9

Equivalent interest rates

t = time in years

i = annual interest rate → change time unit using periods with annual frequency m → ex: $m=12$ months, $m=2$ semesters → t_m = time in periods with annual frequency m → $m\cdot t$ → ex: $t=1{,}5$ years ↳ $t_{12}=18$ months

↳ i_m = period interest rate equivalent to i → ex: i_{12} = monthly interest rate

↓

means: $f(t)=f(t,i)=f(t_m,i_m)=f(t_m)$ for every t and $t_m=m\cdot t$

- **simple**: $f(t)=1+it=1+i_m\cdot mt=f(t_m)$ $\forall t$

$\rightarrow i=i_m\cdot m \leftrightarrow i_m=\frac{i}{m}$

ex: $i=12\%$ simple $\rightarrow i_{12}=\frac{12\%}{12}=1\%$ simple

$i_4=4{,}1\%$ (3 months period) simple interest rate $\rightarrow i=4\cdot 4{,}1\%=16{,}4\%$ simple per year

- **compound**: $(1+t)^t=(1+i_m)^{mt} \rightarrow 1+i=(1+i_m)^m$
  - $i=(1+i_m)^m-1$ period → annual
  - $i_m=(1+i)^{1/m}-1$ annual → period

ex: $i=12\%$ compound $\rightarrow i_{12}=\sqrt{1{,}12}-1\sim 0{,}0583 \rightarrow 5{,}83\%<6$ compound

$i_{12}=1\%$ compound $\rightarrow i=1{,}01^{12}-1\sim 0{,}1268 \rightarrow 12{,}86\%$ compound

- compound (m-convertible): nominal annual rate (TAN)

$J_m=m\cdot i_m$ where i_m is the compound period rate

ex: $i_{12}=1\%$ compound
- $i=12{,}68$
- $J_{12}=12\cdot 1\%=12\%$ → $J_m<i$ always

nominal (TAN)

effective (TAE, TAEG, TEG)

- compound: force of interest δ → in general: let $f:\mathbb{R}_+\rightarrow\mathbb{R}$ a derivable accomulation factor → the force of interest for f is

$\delta=\ln(1+i)$ annual $\rightarrow (1+i)^t=e^{\delta t}$ $\forall t\geq 0 \rightarrow i=e^{\delta}-1$

$$\delta(t)=\frac{f'(t)}{f(t)}=\frac{d\ln f(t)}{dt}$$

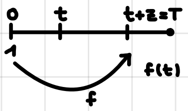
→ex: $f(t)=1+it \rightarrow f'(t)=i$ $\forall t \rightarrow \delta(t)=\frac{i}{1+it}$

N.B.: if $f'(t)$ is continuous then

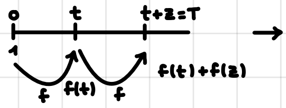
$$f(t)=e^{\int_0^t \delta(s)\,ds}$$

Decomposability

Investment lifetime $T=t+z$ → $C=1$, consider the 2 strategies:



or



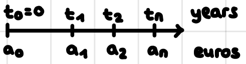
→ an accomulation factor f is said decomposable if $f(t+z)=f(t)\cdot f(z)$ for all $t,z\geq 0$

→ $f(t)=(1+i)^t=e^{\delta t}$ is the unique f(t) which is decomposable

26 Febbraio 2024 - Lezione 10

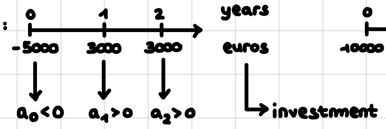
Cash flows

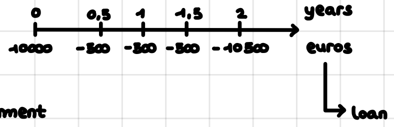
(A)

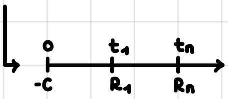


years	0	t_1	t_2	t_n
flows	a_0	a_1	a_2	a_n

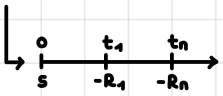
ex:







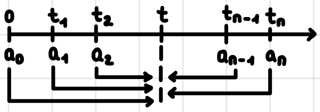
$C>0$ $R_n>0$



in general given an accomulation factor f

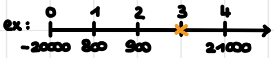
and a discount factor φ, we call value of → $V_A(t)=\underbrace{\sum a_k\cdot f(t-t_k)}_{t_k\leq t}+\underbrace{\sum a_k\cdot \varphi(t_k-t)}_{t_k>t}$

a cashflow (A) at the date t



- if $t=0$ → $V(0)=\sum_{k=1}^{n}a_k\cdot\varphi(t_k)$ present value of (A)
- if $t\geq t_k$ → $V(t)=\sum_{k=1}^{n}a_k\cdot f(t-t_k)$ final value at t of (A)

ex:



$i=5\%$ simple interest rate

$$\rightarrow\begin{cases} f(t)=1+0{,}05t \\ \varphi(t)=\dfrac{1}{1+0{,}05t}\end{cases}\longrightarrow$$

$$V(0)=-20000+\frac{800}{1+0{,}05\cdot 1}+\frac{900}{1+0{,}05\cdot 2}+\frac{21000}{1+0{,}05\cdot 4}\sim -919{,}91$$

$$V(t=4)=-20000(1+0{,}05\cdot 4)+800(1+0{,}05\cdot 3)+900(1+0{,}05\cdot 2)+21000\sim -1090$$

$$V(2)=-20000(1+0{,}05\cdot 2)+800(1+0{,}05\cdot 1)+900+\frac{21000}{1+0{,}05\cdot 2}\sim -1169{,}09$$

N.B. usually $f(t)=(1+i)^t$ and $\varphi(t)=(1+i)^{-t}$

compound interest: $V(t)=\sum a_k\cdot(1+i)^{t-t_k}+\sum a_k(1+i)^{-(t_k-t)}$

$$\rightarrow \sum_{k=1}^{n}a_k(1+i)^{t-t_k}=(1+i)^t\cdot\underbrace{\sum_{k=1}^{n}a_k\cdot(1+i)^{-t_k}}_{V(0)}\longrightarrow V(t)=V(0)\cdot(1+i)^t$$

present value V(0) is the "king"

Discounted cash flow (DCF)

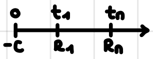
The $V_A(0)=\sum_{1}^{n}a_k(1+x)^{-t_k}$ depends on the compound interest rate x

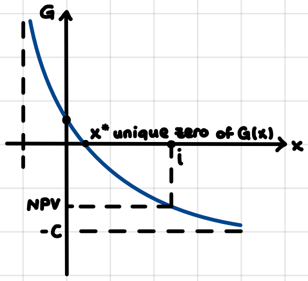
→ the DFC of (A) is the function $G:(-1,+\infty)\rightarrow\mathbb{R}$ defined as

$G(x)=\sum_{k=1}^{n}a_k(1+x)^{-t_k}$ $x>-1$

→ typical cases:

- investment





$G(0)=\sum a_k=-C+\sum R_k$ → accounting balance

$x\rightarrow+\infty$ entails

$G(x)\rightarrow a_0=-C$

G decreasing and

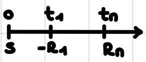
def:

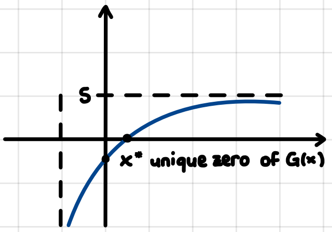
- net present value (NPV) at the compound interest rate i is the value of G at $x=i$
- internal rate of (A) is the compound interest rate x^* such that $G(x^*)=0$

↓

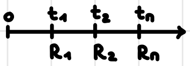
- investment → x^* = internal rate of return (IRR)
- loan → x^* = tasso annuo effettivo di finanziamento (TAE) / TAEG — TEG

- loans





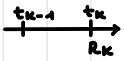
Annuity



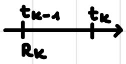
$R_k>0$ → periodic: $t_k-t_{k-1}=p$ constant → $m=\frac{1}{p}$ → constant installments: $R_k=R$, $k=1,\dots n$

p=1 (annual) annuity

- ordinary if the installment R_k of $k-t_k$ period has maturity at the end of the period



- due if

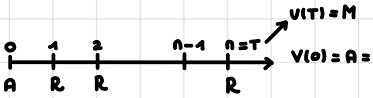


27 Febbraio 2024 - lezione 11

Annuities with constant

i = compound interest rate

- **ordinary**: $t_K = K,\ K = 1, \dots n$



$V(0) = A = \frac{R}{1+i} + \frac{R}{(1+i)^2} + \dots \frac{R}{(1+i)^n} \rightarrow \boxed{V(0) = A = R \cdot \frac{1-(1+i)^{-n}}{i}}$ $a_{\overline{n}|i}$ = a figured n at late i

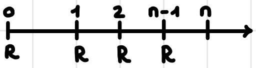
ex: $R = 12000$ $n = 10$ $i = 5\%$ compound $\rightarrow A = 12000 \cdot \frac{1-1{,}05^{-10}}{0{,}05} = 92660{,}82 < 120000$ → you pay the discount

$V(T=n) = M = R(1+i)^{n-1} + R(1+i)^{n-2} + \dots R$

$\rightarrow V(T=n) = M = \underset{\downarrow\ V(0)}{A} \cdot (1+i)^n \rightarrow \boxed{M = R \cdot \frac{(1+i)^n - 1}{i}}$

ex: $V(n) = M = 92660{,}82 \cdot 1{,}05^{10} \sim 150934{,}71 > 12000$ → you get interest

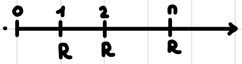
- **due**:



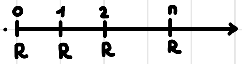
$A = R \cdot a_{\overline{n}|i} \cdot (1+i) \rightarrow \ddot{a}_{\overline{n}|i}$ = a due, figured n at late i

ex: $R = 12000$ $n = 10$ $i = 5\%$ $\rightarrow A = 12000 \cdot \frac{1-1{,}05^{10}}{0{,}05} \cdot 1{,}05 \sim 97293{,}86 \rightarrow M = R \cdot \ddot{a}_{\overline{n}|i}(1+i)^n = R \cdot \overset{\text{final value}}{s_{\overline{n}|i}} \cdot (1+i) = R \cdot \frac{(1+i)^n - 1}{i} \cdot (1+i) = 97293{,}86 \cdot 1{,}05^{10} \sim 158481{,}446$

Perpetuities



there is only the present value A $\rightarrow a_{\overline{n}|i} = \frac{1-(1+i)^{-n \rightarrow \infty}}{i} \rightarrow \boxed{a_{\overline{\infty}|i} = \frac{1}{i}}$



$\ddot{a}_{\overline{n}|i} = \frac{1-(1+i)^n}{i} \cdot (1+i) \rightarrow \ddot{a}_{\overline{\infty}|i} = \frac{1}{i}(1+i) = \boxed{1 + \frac{1}{i}}$

→ ex: $R = 12000$ $i = 5\%$ compound $\rightarrow A = \frac{12000}{0{,}05} = 240000$

$\rightarrow A = 240000 + 12000 = 252000$

→ what if the installments are periodic m times x year?

same formulas with $\boxed{i_m = (1+i)^{1/m} - 1}$ and $\boxed{N = n \cdot m}$ → number of installments per year

number of years

→ ex: $m = 12$ $R = 12000$ $n = 10$ $i = 5\%$

$N = 120$ installments $i_{12} = 1{,}05^{1/12}$ $\rightarrow A = R \cdot a_{\overline{N}|i_m}$

$\rightarrow A = \frac{1-1{,}05^{-10}}{1{,}05^{1/12} - 1} \sim 94765{,}59$ ordinary

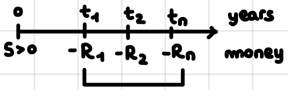
$\rightarrow A = R \cdot a_{\overline{N}|i_m} \cdot (1 + i_m)$

29 Febbraio 2024 - lezione 12

Amortization of a loan

$S > 0$: amount of the loan

$T > 0$: duratin of the loan → pay back bothe the initial debt $S > 0$ and the accured interest gradually →



→ R_K = amortization installments

$\hookrightarrow \boxed{R_K = C_K + I_K}$

C_K → principal quota

I_K → interest quota

↓

all quantities involved depend on the financial law f and φ contractually established

→ outstanding (or residual) debt: D_K the outstanding (i.e. unpaid) debt at t_K^+ after the payment → the amortization is over whenever $D_n = 0$ →

$R_K = C_K + I_K$

$\begin{cases} D_0 = S \\ D_K = D_{K-1} - C_K \end{cases}$ ↳ recursively $\rightarrow \boxed{D_K = \sum_{j=K+1}^{n} C_j}$ at time t_K^+

closure condition:

- elementary $C_K > 0$ with $\boxed{\sum_{K=1}^{n} C_K = S} \leftrightarrow D_n = 0$
- financial given $f(t)$ and $\varphi(t)$
  - → initial: $\boxed{\sum_{K=1}^{n} R_K \cdot \varphi(t_K) = S}$
  - → final: $\boxed{\sum_{K=1}^{n} R_K \cdot f(t_m - t_n) = S \cdot f(t_n)}$

→ from financial closure condition we get:

$\begin{cases} D_0 = S \\ D_K = D_{K-1}(1+i)^{t_K - t_{K-1}} \end{cases} \rightarrow \boxed{D_K = \sum_{j=K+1}^{n} R_j (1+i)^{-(t_j - t_n)}}$

→ if $f(t) = (1+i)^t$ and $\varphi(t) = (1+i)^{-t}$ then

$\boxed{\sum C_K = S \leftrightarrow \sum R_K \cdot (1+i)^{-t_K} = S \leftrightarrow \sum R \cdot (1+i)^{T-t_K} = S(1+i)^{T-t_K}}$

→ if $t_K = K = 1, \dots n$ then:

$\begin{cases} D_0 = S \\ D_K = D_{K-1}(1+i) - R_K \end{cases} \rightarrow \boxed{D_K = \sum_{j=K+1}^{n} R_j (1+i)^{j-K}}$ amortization plan

uniform amortization: $C_K = C \rightarrow C = \frac{S}{n}$

ex: $S = 200000$ $n = 4$

$i = 6\%$ compound

$t_K = K = 1, \dots 4$

t_K	R_K	C_K	I_K	D_K
0	—	—	—	200000
1	62000	50000	12000	150000
2	59000	50000	9000	100000
3	56000	50000	6000	50000
4	53000	50000	3000	0

calculation of I_K: $I_K = D_{K-1}\left[(1+i)^{t_K - t_{K-1}} - 1\right]$ → if $t_K = K$: $I_K = D_{K-1} \cdot i$

↳ interest accrued on D_{K-1}

in the time period $[t_{K-1}, t_K]$

→ progressive amortization:

$R_K = R$ for all K with $t_K = K$ annual installment

→ from initial condition we get: $\sum^{n} R(1+i)^{-K} = S$ → $R = \frac{S}{a_{\overline{n}|i}} = \frac{S \cdot i}{1-(1+i)^{-n}}$

↓

$R \cdot a_{\overline{n}|i}$

4 marzo 2024 - lezione 13

Financial choices

Risk-free financial operation: 0, 1, 2, n → a_0, a_1, a_2, a_n

financial goal = more money → W_0 = initial wealth of the decision maker (DM)

T = time horizon for the DM

$W_T = W_T(A)$ = final wealth at T if the DM undertakes A

↓

$\max_{A} W_T(A)$

→ choice criterium: → profitability index = max the index

→ non-profitability index = min the index → def: A choice criterium is rational if leads to choose A^* such that $W_T(A^*) \geq W_T(A)$ for all A → N.B. the DM can always do "nothing" label (N) such a "operation"

↓

internal rate criterium:

- calculate x_A^* such that $DFC_A = G_A \cdot (x_A^*) = 0$ for every (A) under choice
- (A) is an investment = x_A^* internal rate of return (IRR)
- (A) is a loan = x_A^* effective rate of the loan

↓

considering investments (A) the IRR criterium is:

- (A) is better than (B) if $x_A^* > x_B^*$
- (A) is equivalent to (B) if $x_A^* = x_B^*$

→ if (A) is not an investment (or a loan) than:

- maybe (A) has more than one x^* → ex. 0, 1, 2 : −500, 1200, −600 → $x_1^* = 42\%$ $x_2^* = 238\%$
- maybe (A) has no internal rate → ex. 0, 1, 2 : 600, −1000, −600 ∄ x^* such that $G(x^*) = 0$
- even if $\exists!\, x_A^*$ you don't know if it is better a greater or a smaller rate...

↑ does exist a unique

→ in general (also in case of investment) the IRR is not rational

↓

NPV criterium:

- calculate the $NPV_A(i) = \sum_{K=0}^{n} a_K (1+i)^{-t_K}$
- (A) is better than (B) if $NPV_A(i) > NPV_B(i)$
- (A) is equivalent to (B) if $NPV_A(i) = NPV_B(i)$

→ assumption:

- the DM own capital W_0 are sufficient to undertake every A under choice
- the DM own capital W_0 is employed at the compound interest rate i in [0,T]

↳ opportunity-cost of capital (equity)

↳ of self-financing

↓

meaning of $NPV_A(i)$ and its rationality:

- compare (A) with (N):

$-W_T(N)$: 0, T ; $W_0 \xrightarrow{i} W_T(N) = W_0(1+i)^T$

$-W_T(A)$: 0, t_1, $t_n \leq T$; $W_0 + a_0$, a_1, a_n

$(W_0+a_0)(1+i)^T$; $a_1(1+i)^{T-t_1}$

→ $W_A(T) = W_0(1+i)^T + \sum_{K=0}^{n} a_K (1+i)^{T-t}$

↳ $W_0(1+i)^T + W_0(1+i)^T \cdot \sum_{K=1}^{n} a_K (1+i)^{-t_K}$

→ (A) is "profitable" if $W_T(A) > W_T(N)$ that is

$\cancel{W_0(1+i)^T} + (1+i)^T \cdot NPV_A(i) > \cancel{W_0(1+i)^T}$

↳ >0

→ $NPV_A(i) > 0$ → $NPV_A(i) = \frac{W_T(A) - W_T(N)}{(1+i)^T}$ → present value of the extra profit : net present value

→ extra profit : net profit

5 marzo 2024 - lezione 14

Adjusted present value (APV)

- Internal rate x_A^*
- $NPV_A(i)$

↳ opportunity cost of capital

→ what if W_A is not enough to undertake A?

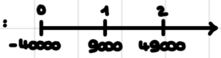
→ ex. (A) 0, 1, 2 : −100000, 15000, 115000 but $W_0 = 40000$

C

(F) 0, 1, 2 : 60000, −6000, −66000 J = 10% → J≠i : the DM may invest money at i = 5%

↳ financing at J = 10% ≠ i

→ex.(continued):



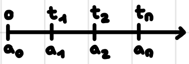
$$APV(5\%) = -40000 + \frac{9000}{1{,}05} + \frac{19000}{1{,}05^2} \sim 13015{,}87\text{ €}$$

remarks:

- $APV_{A+F}(i) = APV_A(i) + APV_F(i)$
- what if $J = i$? → $APV_{A+F}(i) = APV_A(i) + 0$
- to calculate the APV is not necessary to know J

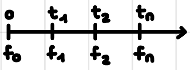
→ in general:

(A)



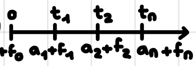
\+

(F)



=

(A+F)



net flows

→ calculate the NPV on equity

$$NPV_{A+F}(i) = \sum_{K=1}^{n} (a_K + f_K)(1+i)^{-t_K}$$

adjusted present value of A+F

at the opportunity cost of capital i → APV(i) profitability index

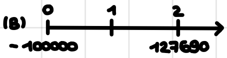
ex: $i = 10\%$ $T = 2$ years

(A)

$x_A^* = \frac{115000}{100000} - 1 = 15\%$ → $NPV_A(0,1) = -1000000 + \frac{115000}{1{,}1} \sim 4545{,}45$

→ $W_{T=2}(A) < W_{T=2}(B)$

(B)



$x_B^* = \left(\frac{127690}{100000}\right)^{1/2} - 1 = 13\%$ → $NPV_B(0,1) = -1000000 + \frac{127690}{1{,}1^2} \sim 5528{,}93$

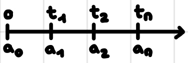
↓ IRR A better than B

↓ NPV B better than A

review exercises:

- how to calculate the internal rate x^*?

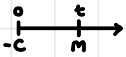
(A)



consider DCF of A → particular case:

and solve $G(x) = 0$

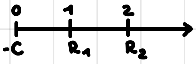
for $x > -1$



$G(x) = -C + \frac{M}{(1+x)^t} = 0$

→ $x^* = \left(\frac{M}{C}\right)^{1/t} - 1$

(B)



$G(x) = a_0 + \frac{a_1}{1+x} + \frac{a_2}{(1+x)^2} = 0$

$u = 1 + x$ → $a_0 u^2 + u a_1 + a_2 = 0$ → $x^* = u^* - 1$

Blab
UNILAB NETWORK

In collaborazione con:

DELIVERY VALLEY
NO GENDER KITCHEN

700+
CLUB

Per dubbi o suggerimenti sulla dispensa:

Elena Cacioli

+39 392 8931605

@elenacaciolii_

Albino Trapuzzano

+39 334 2311588

@a_trapu

Per info sull'Area Didattica:

GABRIELE CARDINALE

+39 320 2126708

@kings_gabb

CHIARA TUA

+39 347 9789059

@chiara_tua